The Langlands correspondence

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21 June 2023

PhD Seminar on local Langlands for GL_2, Talk 12

The main reference is [BH]: for more precise references, see the seminar programme. We claim no originality, except for mistakes, typos, glitches, and errors, which are indeed our very own.

The talk is structured as follows. After making some recollections, we will state the main result of this seminar, namely the Langlands correspondence. This being the titular statement of the seminar, it is our sincere hope that up to this point, no one in the audience will be lost. The rest of the talk is mostly about the proof, and thus will get a bit technical at certain points.

As one may expect, the non-cuspidal part of the correspondence is relatively easy, and we will prove it forthwith. For the cuspidal part, there are two independent approaches. The first one is based on the parametrisation theorem, and turns out to be relatively explicit, but we will only indicate the main idea here. We will say more on the second one; some buzzwords include Bruhat decomposition, Weyl element and Weil representation. Both of these approaches fall short from giving the full picture for $p = 2$, but this should not be surprising, since the ingredients going into them also fail in certain dyadic cases, with ramification being the chief antagonist.

Finally, if time permits, we intend to say a few words on what happens if one considers representations over $\mathbb{Q}_\ell^{\mathrm{c}}$ instead of $\mathbb{C}$ when $\ell \neq p$. In short, things actually work out fine.

0 Recollections

As always, F is a nonarchimedean local field of residue characteristic p. Once and for all we fix an algebraic closure F^{c} of F. We use the symbol $\uplus$ to denote disjoint union.

0.1 The Weil group

We refer back to talk 10. We have the following tower of extensions:

$$\begin{tikzcd}
F^{\mathrm{c}} \arrow[d, dash, "\mathcal{P}_F"] \arrow[dddd, dash, bend right=60, "\Omega_F"'] \arrow[dd, dash, bend right=40, "\mathcal{I}_F"'] & \\
E_\infty \arrow[d, dash] \arrow[dd, dash, bend left=50, "\varprojlim_n \mu_n \simeq \prod_{\ell \neq p} \mathbb{Z}_\ell"] & \text{maximal tamely ramified extension of } F_\infty \\
E_n \arrow[d, dash, "\mu_n"] & \\
F_\infty \arrow[d, dash] \arrow[dd, dash, bend right=40, "\widehat{\mathbb{Z}}"'] & \text{maximal unramified extension of } F \\
F_m \arrow[d, dash, "\mathbb{Z}/m"] & \\
F &
\end{tikzcd}$$

The Weil group is defined by the following diagram whose rows are exact sequences. The topology on the Weil group is as discussed in Talk 10.

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{I}_F \arrow[r] & \Omega_F \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & 0 \\
1 \arrow[r] & \mathcal{I}_F \arrow[r] \arrow[u, equal] & \mathcal{W}_F \arrow[r] \arrow[u, hook] & \mathbb{Z} \simeq \langle \Phi_F \rangle \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

Here the element Φ_F is the geometric Frobenius, the image of which in each finite extension F_m/F is the inverse of the arithmetic Frobenius $x \mapsto x^q$, where q is the order of the residue field of F.

0.2 Deligne representations

A Deligne representation of $\mathcal{W}_F$ is a triple $(\rho, V, \mathfrak{n})$ where (ρ, V) is a finite dimensional smooth representation of $\mathcal{W}_F$, and $\mathfrak{n} \in \mathrm{End}_{\mathbb{C}}(V)$ is a nilpotent endomorphism such that for all $x \in \mathcal{W}_F$,

$$\rho(x)\mathfrak{n}\rho(x)^{-1} = \|x\|\mathfrak{n} \tag{†}$$

The absolute value $\|-\|$ here is the pullback of $|-|_F$ under the Artin map $\mathcal{W}_F \to F^\times$. A Deligne representation is called semisimple if the underlying representation (ρ, V) is. We write

$$\mathcal{G}_2(F) := \{\text{2-dimensional semisimple Deligne representations of } \mathcal{W}_F\}/\simeq$$

We shall repeatedly abuse notation and talk about representations in $\mathcal{G}_2(F)$ instead of their equivalence classes.

1 Statement and first steps

Recall the notation

$$\mathcal{A}_2(F) := \{\text{smooth irreducible representations of } G = \mathrm{GL}_2(F)\}/\simeq$$

We are ready to state the Langlands correspondence.

Theorem 1 (Langlands correspondence, [BH, §33.1]). *Let $\psi \in \widehat{F}$, $\psi \neq \mathbb{1}$. Then there is a unique map*

$$\boldsymbol{\pi} : \mathcal{G}_2(F) \to \mathcal{A}_2(F)$$

such that for all $\rho \in \mathcal{G}_2(F)$ and for all $\chi \in \widehat{F}^\times$:

$$L(\chi \otimes \rho, s) = L(\chi\boldsymbol{\pi}(\rho), s)$$
$$\varepsilon(\chi \otimes \rho, s, \psi) = \varepsilon(\chi\boldsymbol{\pi}(\rho), s, \psi)$$

Moreover, $\boldsymbol{\pi}$ is a bijection, and the equations for the L-factors and ε-factors above hold for all $\psi \neq \mathbb{1}$, i.e. not just the one with which we started out. ○

Note that uniqueness is automatic from the converse theorem (talks 8 & 9): indeed, this tells us that L and ε determine representations up to isomorphism. A recurring theme of this seminar is distinguishing between non-cuspidal and cuspidal representations, with the latter being significantly more difficult. (Just recall that we spent one talk on the classification of the former and three on the latter, or think of the proof of the converse theorem.) This is once again what happens here: we take decompositions $\mathcal{G}_2(F) = \mathcal{G}_2^0(F) \uplus \mathcal{G}_2^1(F)$ resp. $\mathcal{A}_2(F) = \mathcal{A}_2^0(F) \uplus \mathcal{A}_2^1(F)$ where

$$\mathcal{G}_2^0(F) := \{\text{2-dimensional } \textbf{irreducible} \text{ smooth Deligne representations of } \mathcal{W}_F\}/\simeq$$
$$\mathcal{G}_2^1(F) := \{\text{2-dimensional } \textbf{reducible} \text{ semisimple Deligne representations of } \mathcal{W}_F\}/\simeq$$
$$\mathcal{A}_2^0(F) := \{\text{smooth irreducible } \textbf{cuspidal} \text{ representations of } G = \mathrm{GL}_2(F)\}/\simeq$$
$$\mathcal{A}_2^1(F) := \{\text{smooth irreducible } \textbf{non-cuspidal} \text{ representations of } G = \mathrm{GL}_2(F)\}/\simeq$$

The point of this decomposition is the following:

Proposition 2 ([BH, §33.2]). *We have the following characterisation of the subsets $\mathcal{G}_2^0$ resp. $\mathcal{A}_2^0$:*

$$\mathcal{G}_2^0(F) = \left\{\rho \in \mathcal{G}_2(F) : \forall \chi \in \widehat{F}^\times, L(\chi \otimes \rho, s) = 1\right\}$$
$$\mathcal{A}_2^0(F) = \left\{\pi \in \mathcal{A}_2(F) : \forall \chi \in \widehat{F}^\times, L(\chi\pi, s) = 1\right\}$$

In particular, the Langlands correspondence $\boldsymbol{\pi}$, if it exists, must consist of maps

$$\boldsymbol{\pi}|_{\mathcal{G}_2^0(F)} : \mathcal{G}_2^0(F) \to \mathcal{A}_2^0(F) \quad \text{and} \quad \boldsymbol{\pi}|_{\mathcal{G}_2^1(F)} : \mathcal{G}_2^1(F) \to \mathcal{A}_2^1(F)$$ ○

Proof. We have seen the first assertion in talk 8 and the second one in talk 10. □

In line with our previous experience, the non-cuspidal part of the correpondence is easy. It is given as follows.

Theorem 3 ([BH, §33.3])**.** *There is a unique map*

$$\boldsymbol{\pi}^1 : \mathcal{G}_2^1(F) \to \mathcal{A}_2^1(F)$$

such that for all $\rho \in \mathcal{G}_2^1(F)$ *and for all* $\chi \in \widehat{F}^\times$*,*

$$L(\chi \otimes \rho, s) = L(\chi \boldsymbol{\pi}^1(\rho), s)$$

Moreover, $\boldsymbol{\pi}^1$ *is a bijection, and for all* $\rho \in \mathcal{G}_2^1(F)$ *and for all* $\chi \in \widehat{F}^\times$*, and every nontrivial* $\psi \in \widehat{F}$*,*

$$\boldsymbol{\pi}^1(\chi \otimes \rho) = \chi \boldsymbol{\pi}^1(\rho)$$
$$\varepsilon(\rho, s, \psi) = \varepsilon(\boldsymbol{\pi}^1(\rho), s, \psi)$$

∘

Proof. We will define the map $\boldsymbol{\pi}^1$ explicitly. Let $(\rho, V, \mathfrak{n}) \in \mathcal{G}_2^1(F)$ be a 2-dimensional reducible Deligne representation of $\mathcal{W}_F$. Since ρ is semisimple, we have

$$\rho = \chi_1 \oplus \chi_2$$

for some characters $\chi_1, \chi_2 \in \widehat{F}^\times$. Define a character on the diagonal torus as

$$\chi := \chi_1 \otimes \chi_2 \in \widehat{T}, \quad \operatorname{diag}(t_1, t_2) \mapsto \chi_1(t_1)\chi_2(t_2)$$

We can take its normalised induction to G:

$$\pi := \boldsymbol{\iota}_B^G \chi$$

Recall that this is $\operatorname{Ind}_B^G(\delta_B^{-1/2} \otimes \chi)$ where $\delta_B : B = TN \to \mathbb{C}$, $tn \mapsto \|t_2/t_1\|$. Furthermore, in talk 4 we have seen that π is reducible iff $\chi_1\chi_2(x) = \|x\|^{\pm 1}$ iff $\chi = \varphi \delta_B^{\pm 1/2}$ for some $\varphi \in \widehat{F}^\times$. We distinguish between two cases based on whether χ is irreducible, and then apply these equivalences.

If π is irreducible, then $\mathfrak{n} = 0$. Indeed, writing $\mathfrak{n} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, condition (†) reads:

$$(\chi_1 \oplus \chi_2)(x)\mathfrak{n}(\chi_1 \oplus \chi_2)(x)^{-1}\|x\|\mathfrak{n}$$
$$\begin{pmatrix} a & \chi_1(x)\chi_2(x)^{-1}b \\ \chi_1(x)^{-1}\chi_2(x) & d \end{pmatrix} = \|x\| \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Since $\chi_1\chi_2(x) \neq \|x\|^{\pm 1}$, this forces $b = c = 0$. Since $\mathfrak{n}$ is nilpotent, $a = d = 0$ must also hold, thus $\mathfrak{n} = 0$, as claimed.

If π is not irreducible, then $\chi_1(x) = \varphi(x)\|x\|^{-1/2}$ and $\chi_2(x) = \varphi(x)\|x\|^{1/2}$ for some $\varphi \in \widehat{F}^\times$. We claim that there are exactly two Deligne representations $(\rho, V, \mathfrak{n}) \in \mathcal{G}_2^1(F)$ such that ρ has composition factors χ_1 and χ_2. Indeed, if $\mathfrak{n} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then the φ-terms cancel in (†), and we get:

$$\begin{pmatrix} a & \|x\|^{-1}b \\ \|x\|c & d \end{pmatrix} = \begin{pmatrix} \|x\|a & \|x\|b \\ \|x\|c & \|x\|d \end{pmatrix}$$

We immediately conclude that $a = b = d = 0$. Moreover, c is either zero or not, and all Deligne representations with nonzero c are equivalent. Note that while this computation was nice and explicit, we could also have uttered the magic words 'Jordan normal form': a nilpotent matrix $\mathfrak{n}$ will be equivalent to either zero or $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

The definition of $\boldsymbol{\pi}^1$ is as follows:

$$\boldsymbol{\pi}^1((\rho, V, \mathfrak{n})) := \begin{cases} \pi = \boldsymbol{\iota}_B^G \chi & \text{if } \pi \text{ is irreducible} \\ \varphi \circ \det & \text{if } \pi \text{ is not irreducible and } \mathfrak{n} = 0 \\ \varphi \circ \mathrm{St}_G & \text{if } \pi \text{ is not irreducible and } \mathfrak{n} \neq 0 \end{cases}$$

It remains to verify that $\boldsymbol{\pi}^1$ satisfies the prescribed properties. Uniqueness is once again clear from the converse theorem. The rest should follow easily from definitions. □

Therefore it remains to show the following.

Theorem 4 ([BH, §33.4])**.** *Let* $\psi \in \widehat{F}$ *be nontrivial. Then there is a unique map*

$$\boldsymbol{\pi} : \mathcal{G}_2^0(F) \to \mathcal{A}_2^0(F)$$

such that for all $\rho \in \mathcal{G}_2^0(F)$ *and for all* $\chi \in \widehat{F}^\times$*:*

$$\varepsilon(\chi \otimes \rho, s, \psi) = \varepsilon(\chi \boldsymbol{\pi}(\rho), s, \psi)$$

Moreover, $\boldsymbol{\pi}$ *is a bijection, and the equation for* ε*-factors above holds for all* $\psi \neq \mathbb{1}$*, i.e. not just the one with which we started out.*

∘

Note that in comparison with Theorem 1, the equation for L-factors disappeared: this is because of Proposition 2.

Proposition 5 ([BH, §33.4])**.** *If $\boldsymbol{\pi}$ is as in Theorem 4 then:*

(1) *For all $\rho \in \mathcal{G}_2^0$, the central character of $\boldsymbol{\pi}(\rho)$ is $\omega_{\boldsymbol{\pi}(\rho)} = \det \rho$.*

(2) *The equation for ε-factors holds for all $\psi \neq \mathbb{1}$.* ∘

Proof. On either side of the correspondence, there are formulæ concerning ε-factors of twists: on the automorphic side, the central character appears, and on the Galois side, we get the determinant. Since the two ε-factors are supposed to be equal, (1) follows.

The proof of (2) relies on (1) plus some statements we haven't seen. □

2 The tame correspondence

In previous talks, we related admissible pairs to the automorphic as well as the Galois side of the Langlands correspondence. This was made possible by the fact that admissible pairs are relatively easy to work with: indeed, one can study them using local class field theory of quadratic extensions. Since they are related to both sides, one may regard them as mediating objects between the two sides. This can be exploited to derive the full Langlands correspondence in the case $p \neq 2$. Most of the work has already been done in previous talks.

Recall that an admissible pair $(E/F, \xi)$ consists of a tamely ramified quadratic extension E/F and a character $\xi \in \widehat{E}^\times$ such that ξ does not factor through $N_{E/F}$ and if $\xi|_{U_E^1}$ factors through $N_{E/F}$ then E/F is unramified. Under the obvious equivalence relation, the set of equivalence classes is denoted by $\mathbb{P}_2(F)$.

In talks 7 resp. 11, we have seen the following:

Theorem (Parametrisation theorem, [BH, §§20.2 & 34.1])**.** *Let $p \neq 2$. There are bijections*

$$\begin{tikzcd}[column sep=small] & & (E/F,\xi) \arrow[dll] & \in & \mathbb{P}_2(F) \arrow[dl, leftrightarrow, "1:1"'] \arrow[dr, leftrightarrow, "1:1"] & \ni & (E/F,\xi) \arrow[drr] & & \\ \rho_\xi := \operatorname{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F} \xi & \in & & \mathcal{G}_2^0(F) & & \mathcal{A}_2^0(F) & & \ni & \pi_\xi \end{tikzcd} \tag{‡}$$

We omit the definition of π_ξ. It satisfies several properties, including that its central character is $\omega_{\pi_\xi} = \xi|_{F^\times}$. ∘

Remark. Both theorems admit a tame version when $p = 2$, replacing $\mathcal{A}_2^0$ resp. $\mathcal{G}_2^0$ by an appropriate unramified subset, the point being that wild ramification causes problems. We shun the case $p = 2$ from now on. ∘

Putting the two bijections together, we get a correspondence

$$\mathcal{G}_2^0(F) \to \mathcal{A}_2^0(F)$$
$$\rho_\xi \mapsto \pi_\xi$$

This must be the famous Langlands correspondence, right? Wrong! Indeed, on the left hand side, we have that $\det \rho_\xi$ is $\xi|_{F^\times}$ twisted by a nontrivial character: this was a lemma in talk 11. However, on the right, π_ξ has central character $\xi|_{F^\times}$. In particular, $\det \rho \neq \omega_{\pi_\rho}$, contradicting Proposition 5(1). Hence to obtain the Langlands correspondence, one needs to modify this map to account for this twist. This can be done, and we shall say no more about it.

3 The Weil representation

In this section, we will discuss an alternative approach to the Langlands correspondence. According to [BH], the point of having both of these approaches is that certain things are easy in one approach and difficult in the other. Since in this talk, we don't go beyond proving the correspondence for $p \neq 2$ and a partial correspondence for $p = 2$, we won't see any examples of this difference, so we will just have to take their word for it.

3.1 A mirabolic-metaplectic fairytale

Let us recall the central idea from talk 9, which provided the key to proving the converse theorem (on the automorphic side). Let π be an irreducible cuspidal representation of G. Then G acts on Kirillov models $K(\pi, \psi)$

of π, and this can be transferred to an action of G on $C_c^\infty(F^\times)$. We obtain a representation $(\pi_K, C_c^\infty(F^\times))$ such that $\pi \simeq \pi_K$. We have the Bruhat decomposition using the standard Borel B and the Weyl element w:

$$G = B \uplus BwB \quad \text{where} \quad G = \mathrm{GL}_2(F) \supset B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}, w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then π is determined by the central character ω_π and $\pi_K(w)$, and the latter is directly related to ε-constants.

$$\begin{tikzcd}
G = B \uplus BwB \arrow[d, rightsquigarrow] & & & \\
\boxed{G \curvearrowright K(\pi,\psi)} \arrow[r, rightsquigarrow] & \boxed{\pi_K : G \curvearrowright C_c^\infty(F^\times)} \arrow[r, rightsquigarrow] & \pi_K(w) \arrow[r, leftrightarrow] & \varepsilon(\chi\pi, s, \psi) \\
\psi \in \widehat{F}^\times \arrow[u, rightsquigarrow] & & &
\end{tikzcd}$$

We will now recycle these ideas. The battle plan is as follows. Let E/F be a separable quadratic field extension. We shall define a locally profinite group $\mathcal{G}$ with properties similar to those of G: in particular, it will possess a Bruhat decomposition. We will obtain an action of $\mathcal{G}$ on $C_c^\infty(E^\times)$. Denoting the associated representation by $\pi_{E/F}$, we shall study $\pi_{E/F}(\Theta)$ for Θ a character of $E^\times$.

The character Θ is nontrivial on $\mathcal{K} := \mathcal{K}_{E/F} := \ker N_{E/F}$ if and only if $\pi_{E/F}(\Theta)$ is an irreducible cuspidal representation. The Langlands correspondence for $p \neq 2$ is then as follows:

$$\begin{aligned} \mathfrak{G}_2^0(F) &\xrightarrow{\sim} \mathcal{A}_2^0(F) \\ (E/F, \Theta) &\mapsto \pi_{E/F}(\Theta) \end{aligned}$$

For $p = 2$, this only gives a map from a subset of $\mathfrak{G}_2^0(F)$.

Here is a diagrammatic depiction of what we are about to do. The individual steps shall be explained below. We write $N := N_{E/F}$ for the norm map.

$$\begin{tikzcd}
E/F \arrow[d, rightsquigarrow] & \vartheta \in \widehat{\mathcal{K}} \arrow[d, rightsquigarrow] & & & \\
\mathcal{G} = \mathcal{B} \uplus \mathcal{B}w\mathcal{N} \arrow[d, rightsquigarrow, "\text{Fourier}"', "\text{analysis}"] & \Theta \in \widehat{E}^\times \arrow[d, rightsquigarrow] & & & \\
\boxed{\begin{matrix} \eta_\psi^{\mathcal{G}} \\ \mathcal{G} \curvearrowright C_c^\infty(E) \end{matrix}} \arrow[r, rightsquigarrow, "\substack{\text{restrict} \\ \text{twist by } \Theta^{-1}}"'] & \boxed{\begin{matrix} \xi(\Theta,\psi) \\ \mathcal{G} \curvearrowright C_c^\infty(E,\vartheta) \end{matrix}} \arrow[r, rightsquigarrow, "\mathcal{K}\text{-coset invariance}"'] & \boxed{\begin{matrix} \xi_\kappa(\Theta,\psi) \\ G_\kappa \curvearrowright C_c^\infty(N(E^\times)) \end{matrix}} \arrow[r, rightsquigarrow, "\text{twist by } \|-\|_F^{1/2}\Theta(-)"'] & \boxed{\begin{matrix} \pi_\kappa(\Theta,\psi) \\ G_\kappa \curvearrowright C_c^\infty(N(E^\times)) \end{matrix}} \arrow[r, rightsquigarrow, "\mathrm{Ind}_{G_\kappa}^{\mathcal{G}}"'] & \boxed{\begin{matrix} \pi(\Theta,\psi) \\ G \curvearrowright C_c^\infty(N(E^\times)) \end{matrix}} \\
\psi \in \widehat{F} \arrow[u, rightsquigarrow] & & & &
\end{tikzcd}$$

Remark. What does the word metaplectic mean here? There is a much more general theory of Weil representations: this is a certain infinite-dimensional representation of the relevant metaplectic group. ∘

3.2 The group $\mathcal{G}$ and the representation $\eta_\psi^{\mathcal{G}}$

Let E/F be a quadratic separable extension with Galois group $\mathrm{Gal}(E/F) = \{1, \sigma\}$. We define the group $\mathcal{G} = \mathcal{G}_{E/F}$ and exhibit the fact that its subgroup structure bears a strong resemblance to that of G:

$$\begin{aligned}
\mathcal{G} &:= \left\{ (g,h) : g \in G = \mathrm{GL}_2(F), h \in E^\times, \det g = N_{E/F}(h)^{-1} \right\} \subset G \times E^\times \\
\mathcal{K} &:= \ker N_{E/F} \simeq \left\{ (g,h) \in \mathcal{G} : N_{E/F}(h) = 1 \right\} \\
\mathcal{N} &:= \left\{ \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, 1 \right) : t \in F \right\} \\
\mathcal{B} &:= \left(B \times E^\times \right) \cap \mathcal{G}
\end{aligned}$$

The Weyl element for $\mathcal{G}$ is the following; by abuse of notation, it will again be denoted by w:

$$w := (w, 1) = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \in \mathcal{G}$$

This gives rise to the Bruhat decomposition

$$\mathcal{G} = \mathcal{B} \uplus \mathcal{B}w\mathcal{N}$$

Let $\kappa = \kappa_{E/F} \in \widehat{F}^\times$ be the unique character with kernel equal to the norm group $N_{E/F}(E^\times)$. Let $G_\kappa := \ker(\kappa \circ \det) \subset G$, so that there is an exact sequence $1 \to \mathcal{K} \to \mathcal{G} \to G_\kappa \to 1$.

The following theorem is the main technical result in the construction of $\pi_{E/F}(\Theta)$. The proof is long, and involves a lot of Fourier analysis. We shall not concern ourselves with it.

Theorem 6 ([BH, §38.6]). *Let $\psi \in \widehat{F}$ be nontrivial. Then there exists a unique representation $\eta_\psi^{\mathcal{G}}$ of $\mathcal{G}$ on $C_c^\infty(E)$ such that for all $f \in C_c^\infty(E)$ and $y \in E$:*

$$\eta_\psi^{\mathcal{G}}\left(\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}, 1\right)\right) f : y \mapsto \psi\left(tN_{E/F}(y)\right) f(y) \qquad \forall t \in F$$

$$\eta_\psi^{\mathcal{G}}\left(\left(\begin{pmatrix}N_{E/F}(x) & 0\\ 0 & 1\end{pmatrix}, x^{-\sigma}\right)\right) f : y \mapsto \|x\|_E^{1/2} f(xy) \qquad \forall x \in E^\times$$

$$\eta_\psi^{\mathcal{G}}\left(\left(\begin{pmatrix}\zeta & 0\\ 0 & \zeta\end{pmatrix}, \zeta^{-1}\right)\right) f : y \mapsto \kappa(\zeta) f(y) \qquad \forall \zeta \in F^\times$$

$$\eta_\psi^{\mathcal{G}}(w) f : y \mapsto \lambda_{E/F}(\psi)\hat{f}(y^\sigma)$$

Moreover, $\eta_\psi^{\mathcal{G}}$ is smooth. ◦

It is easily seen that every element in $\mathcal{B}$ is the product of elements as in the first three conditions here. This together with the Bruhat decomposition shows that the four conditions determine the representation $\eta_\psi^{\mathcal{G}}$ uniquely on the entire $\mathcal{G}$.

3.3 Constructing the Weil representation

Let $\vartheta \in \widehat{\mathcal{K}}$ be nontrivial. Define a space of functions

$$C_c^\infty(E, \vartheta) := \{f \in C_c^\infty(E) : f(xy) = \vartheta(x) f(y)\, \forall x \in \mathcal{K}, y \in E\}$$

This can be seen as a subspace of $C_c^\infty(E^\times)$ since $f(0) = 0$ because it is constant in a neighbourhood of 0 and ϑ is nontrivial. In fact, it is a $\mathcal{G}$-subspace of $\eta_\psi^{\mathcal{G}}$, that is, the $\mathcal{G}$-action descends to it. This is because $\mathcal{K}$ is central in $\mathcal{G}$ and the condition in the definition of $C_c^\infty(E, \vartheta)$ is equivalent to

$$\eta_\psi^{\mathcal{G}}\left(\left(\begin{pmatrix}N_{E/F}(x) & 0\\ 0 & 1\end{pmatrix}, x^{-\sigma}\right)\right) f = \vartheta(x) f$$

because $\|x\|_E^{1/2} = 1$ for all $x \in \mathcal{K}$. Let $\Theta \in \widehat{E}^\times$ be a lift of ϑ, that is $\Theta|_{\mathcal{K}} = \vartheta$. This allows us to define a representation $\xi(\Theta, \psi) := \Theta^{-1}\eta_\psi^{\mathcal{G}}$ of $\mathcal{G}$ on $C_c^\infty(E, \vartheta)$ as

$$\xi(\Theta, \psi)(g, h) := \Theta(h)^{-1}\eta_\psi^{\mathcal{G}}(g, h)$$

Due to the Θ^{-1}-twisting, this representation is trivial on $\mathcal{K}$. Hence it is the inflation of a $\mathcal{G}/\mathcal{K} = G_\kappa$-representation.

Let $f \in C_c^\infty(E, \vartheta)$. Then $\Theta^{-1} f$ depends only on $\mathcal{K}$-cosets, which shows that there is a linear isomorphism $C_c^\infty(E, \vartheta) \simeq C_c^\infty(N_{E/F}(E^\times))$. We obtain a G_κ-representation $\xi_\kappa(\Theta, \psi)$.

Consider the following linear automorphism of $C_c^\infty(N_{E/F}(E^\times))$:

$$f \mapsto \|-\|_F^{1/2} \cdot \Theta(-) \cdot f(-)$$

Compose $\xi_\kappa(\Theta, \psi)$ with this automorphism, and call this G_κ-representation $\pi_\kappa(\Theta, \psi)$. Finally, let

$$\pi(\Theta, \psi) := \operatorname{Ind}_{G_\kappa}^{G} \pi_\kappa(\Theta, \psi)$$

Remark. A vague idea of what is going on with this construction is the following. We want to study G-representations of the space $C_c^\infty(E)$. Fourier analysis gave us a representation of $\mathcal{G}$ which knows Landlandsy data. A standard idea in representation theory is to decompose a space with respect to characters: this rôle is played by Θ resp. ϑ. To get well-behaved representations, we first have to remove these theta-factors and $\|-\|^{1/2}$: then we can pass from $\mathcal{G}$ to G_κ. We put back the previously removed factors, and induce to G. ◦

Remark. Using Theorem 6, one can explicitly describe the group actions occurring in this section. We don't believe in inundating our audience with a barrage of formulæ, which is why these have been omitted. ◦

3.4 Properties of the Weil representation

Given $\vartheta \in \widehat{\mathcal{K}}$ nontrivial, its lift $\Theta \in \widehat{E}^\times$ is unique up to tensoring with $\chi \circ N_{E/F}$ for a character χ of $N_{E/F}(E^\times)$.

Definition 7. A character $\Theta \in \widehat{E}^\times$ is called *E/F-regular* if $\Theta^\sigma \neq \Theta$. ∘

It is easily seen that $\Theta|_{\mathcal{K}}$ is nontrivial if and only if Θ is E/F-regular. In other words, the characters $\Theta \in \widehat{E}^\times$ arising from $\vartheta \in \widehat{\mathcal{K}}$ as above are precisely the E/F-regular ones.

The main properties of $\pi(\Theta, \psi)$ are summarised as follows.

Theorem 8 ([BH, §§39.2–4])**.** *Let $\Theta \in \widehat{E}^\times$ be an E/F-regular character, and let $\psi \in \widehat{F}$ be nontrivial. Then the following hold.*

(1) *$\pi(\Theta, \psi)$ is an irreducible cuspidal representation of G.*

(2) *For all $\chi \in \widehat{F}^\times$:*
$$\pi\left((\chi \circ N_{E/F})\Theta, \psi\right) \simeq \chi\pi(\Theta, \psi)$$
In particular, for $\chi := \kappa_{E/F}$ we have $\kappa_{E/F} \circ N_{E/F} = \mathbb{1}_{E^\times}$, hence
$$\pi\left(\Theta, \psi\right) \simeq \kappa_{E/F}\pi(\Theta, \psi)$$

(3) *The central character of $\pi(\Theta, \psi)$ is*
$$\omega_{\pi(\Theta,\psi)} = \kappa_{E/F} \otimes \Theta|_{F^\times}$$

(4) *For all $\chi \in \widehat{F}$ nontrivial:*
$$\varepsilon\left(\chi \otimes \operatorname{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F} \Theta, s, \psi\right) = \varepsilon(\chi\pi(\Theta, \psi), s, \psi)$$
Moreover, this equality of ε-factors holds for all ψ (not just the one we started with). Hence $\pi(\Theta, \psi)$ is independent of ψ up to isomorphism. ∘

Proof (sketch). For (1), let $\vartheta \in \widehat{N}$ be defined by $\vartheta(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)) := \psi(x)$. Going through the construction above, it follows that $\pi(\Theta, \psi)|_M = \text{c-}\operatorname{Ind}_N^M \vartheta$. The assertion follows from a result in talk 9.

Formulæ (2) and (3) follow from the construction of $\pi(\Theta, \psi)$.

Verifying the equality of ε-factors in (4) can be reduced to doing so for $\chi = \mathbb{1}$ by (2). Now one needs to do actual computations, going back to the definition of ε-factors, and involving Fourier analysis. The assertion that the equality holds for all characters follows from previous results on ε-factors. The last assertion is a consequence of the converse theorem. □

Definition ([BH, §39.4])**.** The representation $\pi(\Theta) := \pi(\Theta, \psi)$ is called the *Weil representation* defined by the pair $(E/F, \Theta)$. ∘

3.5 Establishing the correspondence

Theorem 8 asserts that the Weil representation $\pi(\Theta)$ possesses all properties required for the Langlands correspondence for induced characters, i.e. when $\rho = \operatorname{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F} \Theta$.

Definition ([BH, §40.1])**.** A representation $\rho \in \mathcal{G}_2^0(F)$ is called *imprimitive* if there exists a separable quadratic extension E/F and a character $\xi \in \widehat{E}^\times$ such that $\rho \simeq \operatorname{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F} \xi$. ∘

As we have seen in talk 11, every class in $\mathcal{G}_2^0(F)$ is imprimitive, see (‡) whenever $p \neq 2$. This together with Theorem 8 establishes the Langlands correspondence for $p \neq 2$. □

4 Dyadic Langlands correspondence

For $p = 2$, there are strict containments $\mathcal{G}_2^{\text{nr}}(F) \subsetneq \mathcal{G}_2^{\text{im}}(F) \subsetneq \mathcal{G}_2^0(F)$. In other words, we also need to deal with primitive representations. As explained in a lemma in talk 10, an irreducible representation of the Weil group $\mathcal{W}_F$ can be seen as a representation ρ of the absolute Galois group Ω_F. Consider the image of ρ in $\mathrm{PGL}_2(\mathbb{C})$:
$$1 \to \rho(\Omega_F) \cap \mathbb{C}^\times \to \rho(\Omega_F) \to \overline{\rho(\Omega_F)} \to 1$$
Using local class field theory for $p = 2$, one can show that $\overline{\rho(\Omega_F)}$ is a finite group. Classification of Platonic solids (well, not really, but something like that) yields that finite subgroups of $\mathrm{PGL}_2(\mathbb{C})$ are is either cyclic, dihedral, the alternating groups A_4 or A_5, or the symmetric group S_4. The cyclic case is when ρ is reducible, and the dihedral one is when ρ is induced. For $p \neq 2$, this is everything, but for $p = 2$, the cases A_4 resp. S_4 also make an appearance: these are called tetrahedral resp. octahedral. The group A_5 is excluded because of a solubility condition.

5 The ℓ-adic correspondence

Let $\ell \neq p$, and let $\mathbb{Q}_\ell^c$ denote an algebraic closure of $\mathbb{Q}_\ell$. The point of this final section is to transfer the Langlands correspondence from complex representations to $\mathbb{Q}_\ell$. For this, we need to introduce appropriate modifications of $\mathcal{G}$ resp. $\boldsymbol{\mathcal{A}}$:

$$\begin{aligned}\mathcal{G}_2(F, \mathbb{Q}_\ell^c) &:= \{\text{2-dimensional semisimple Deligne representations of } \mathcal{W}_F \text{ over } \mathbb{Q}_\ell^c\}/\simeq \\ \boldsymbol{\mathcal{A}}_2(F, \mathbb{Q}_\ell^c) &:= \{\text{smooth irreducible representations of } G = \mathrm{GL}_2(F) \text{ over } \mathbb{Q}_\ell^c\}/\simeq\end{aligned}$$

Note that in this ℓ-adic setting, some of the tools from before can no longer be used: for instance, the normalised induction functor makes no sense, because it involves a well-defined square root, which is not available in the ℓ-adic world. Hence our strategy is to try and transport the complex results directly, hoping that enough of the machinery remains valid.

Choose an isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \mathbb{Q}_\ell^c$. This induces bijections:

$$\begin{tikzcd} \sigma \arrow[d, mapsto] & \mathcal{G}_2(F) \arrow[r, "1:1", "\text{Theorem 1}"'] \arrow[d, "1:1"'] & \boldsymbol{\mathcal{A}}_2(F) \arrow[d, "1:1"] & \pi \arrow[d, mapsto] \\ {}^\iota\sigma & \mathcal{G}_2(F, \mathbb{Q}_\ell^c) \arrow[r, dashed] & \boldsymbol{\mathcal{A}}_2(F, \mathbb{Q}_\ell^c) & {}^\iota\pi \end{tikzcd}$$

But this is not canonical: the automorphism group $\operatorname{Aut}\mathbb{C}$ acts on $\mathcal{G}_2(F)$ and $\boldsymbol{\mathcal{A}}_2(F)$, but this action does not respect the dashed arrow.

Definition ([BH, §35.1]). Let $\rho \in \mathcal{G}_2(F)$ be semisimple. Let $\tilde{\rho}(x) := \|x\|^{-1/2}\rho(x)$: then $\tilde{\rho} \in \mathcal{G}_2(F)$ is semisimple. For a Deligne representation $\sigma = (\rho, V, \mathfrak{n})$, let $\tilde{\sigma} := (\tilde{\rho}, V, \mathfrak{n})$. Finally, define a map

$$\begin{aligned}\Pi_\mathbb{C} : \mathcal{G}_2(F) &\to \boldsymbol{\mathcal{A}}_2(F) \\ \sigma &\mapsto \boldsymbol{\pi}(\tilde{\sigma})\end{aligned}$$

○

Theorem 9 ([BH, §35.1]). *The map $\Pi_\mathbb{C}$ is a bijection, and it commutes with the action of* $\operatorname{Aut}\mathbb{C}$*: for all $\rho \in \mathcal{G}_2(F)$ and $\varphi \in \operatorname{Aut}\mathbb{C}$:*

$$\Pi_\mathbb{C}({}^\varphi\rho) = {}^\varphi\Pi_\mathbb{C}(\rho)$$

Moreover, $\Pi_\mathbb{C}$ is the unique map $\mathcal{G}_2(F) \to \boldsymbol{\mathcal{A}}_2(F)$ such that for all $\chi \in \widehat{F}^\times$, $\psi \in \widehat{F}$ nontrivial, and $\sigma \in \mathcal{G}_2(F)$:

$$\begin{aligned}L\left(\chi \otimes \sigma, s - \frac{1}{2}\right) &= L(\chi\Pi_\mathbb{C}(\sigma), s) \\ \varepsilon\left(\chi \otimes \sigma, s - \frac{1}{2}, \psi\right) &= \varepsilon(\chi\Pi_\mathbb{C}(\sigma), s, \psi)\end{aligned}$$

○

An immediate consequence is the following:

Corollary 10 (ℓ-adic Langlands correspondence, [BH, §35.1]). *There is a unique bijection*

$$\Pi_\ell : \mathcal{G}_2(F, \mathbb{Q}_\ell^c) \to \boldsymbol{\mathcal{A}}_2(F, \mathbb{Q}_\ell^c)$$

such that for all $\sigma \in \mathcal{G}_2(F)$ and $\iota : \mathbb{Q}_\ell^c \xrightarrow{\sim} \mathbb{C}$:

$$\Pi_\ell({}^\iota\sigma) = {}^\iota\Pi_\mathbb{C}(\sigma)$$

○

That is, the following commutative diagram respects the action of $\operatorname{Aut}\mathbb{C}$:

$$\begin{tikzcd} \sigma \arrow[d, mapsto] & \mathcal{G}_2(F) \arrow[r, "1:1", "\Pi_\mathbb{C}"'] \arrow[d, "1:1"'] & \boldsymbol{\mathcal{A}}_2(F) \arrow[d, "1:1"] & \pi \arrow[d, mapsto] \\ {}^\iota\sigma & \mathcal{G}_2(F, \mathbb{Q}_\ell^c) \arrow[r, "1:1", "\Pi_\ell"'] & \boldsymbol{\mathcal{A}}_2(F, \mathbb{Q}_\ell^c) & {}^\iota\pi \end{tikzcd}$$

The proof of the theorem is by studying how $\operatorname{Aut}\mathbb{C}$ changes the functional equation for GL_1. Some of this is tricky: in the study of Fourier transforms, one has to take into account the fact that the Haar measure now takes values in $\mathbb{Q} \cup \sqrt{q}\mathbb{Q}$ instead of $\mathbb{Q}$ on compact opens.

References

[BH] Colin J. Bushnell and Guy Henniart. *The Local Langlands Conjecture for* GL(2). Grundlehren der mathematischen Wissenschaften 335. Springer-Verlag, 2006.