Signed Selmer groups of supersingular elliptic curves over weakly ramified base fields

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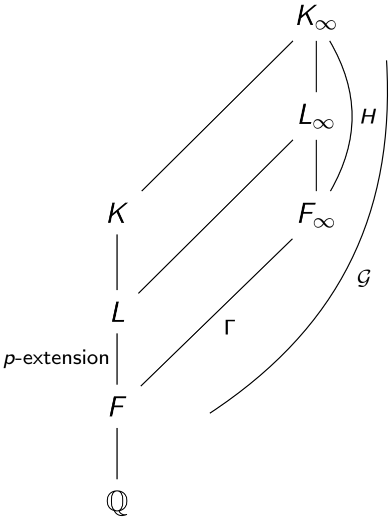
Standing assumptions:

$p \geq 5$ prime

E/F elliptic curve



F_∞/F — cyclotomic $\mathbb{Z}_p$-extension, i.e. the unique quotient extension of $F\mathbb{Q}(\mu_{p^\infty})/F$ with $\Gamma := \mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p$

Main results (rough form)

Kida formula

$F/\mathbb{Q}$ — finite, $K/L/F$ — finite Galois p-extensions
Under suitable assumptions:

$$\lambda\left(X^{\vec{s}}(E/K_\infty)\right) = [K_\infty : L_\infty]\cdot\lambda\left(X^{\vec{s}}(E/L_\infty)\right) + (\text{fudge factor})$$
$$\mu\left(X^{\vec{s}}(E/K_\infty)\right) = [K_\infty : L_\infty]\cdot\mu\left(X^{\vec{s}}(E/L_\infty)\right)$$

Integrality of characteristic elements

Under suitable assumptions: if ξ is a characteristic element for $X^{\vec{s}}(E/K_\infty)$, then for every maximal order $\mathfrak{M} \subset \mathcal{Q}(\mathcal{G}) := \mathrm{Quot}(\Lambda(\mathcal{G}))$:

$$\xi \in \mathrm{im}\left(\mathfrak{M} \cap \mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G}))\right)$$

Assumptions:

$F/\mathbb{Q}$ — finite extension

Σ — finite set of places of F containing p-adic, infinite, and bad places

E/F — elliptic curve with good ordinary reduction at primes above p:
i.e. $\forall v \mid p$: $\widetilde{E}$ is an elliptic curve, $a_v := 1 + p - \#\widetilde{E}(\mathcal{O}_{F_v}/v) \not\equiv 0 \ (p)$.

$$\mathrm{Sel}_{p^\infty}(E/F) := \ker\left(H^1(F, E[p^\infty]) \xrightarrow{\bigoplus \mathrm{res}_v} \bigoplus_{v\in\Sigma} H^1(F_v, E)\right) =$$

$$\ker\left(H^1(F, E[p^\infty]) \xrightarrow{\bigoplus \mathrm{res}_v} \bigoplus_{\substack{v\in\Sigma \\ v\nmid p}} H^1(F_v, E[p^\infty]) \oplus \bigoplus_{\substack{v\in\Sigma \\ v\mid p}} \frac{H^1(F_v, E[p^\infty])}{E(F_v)\otimes \mathbb{Q}_p/\mathbb{Z}_p}\right)$$

$$=: \ker\left(H^1(F, E[p^\infty]) \xrightarrow{\bigoplus \mathrm{res}_v} \bigoplus_{v\in\Sigma} J_v(E/F_v)\right)$$

F_∞/F — cyclotomic $\mathbb{Z}_p$-extension

F_n/F — the unique degree p^n extension inside F_∞

$\Lambda := \mathbb{Z}_p[\![\Gamma]\!] \simeq \mathbb{Z}_p[[T]]$ Iwasawa algebra

$\mathrm{Sel}(E/F_\infty) := \varinjlim_n \mathrm{Sel}_{p^\infty}(E/F_n)$ discrete Λ-module

$X(E/F_\infty) := \mathrm{Hom}(\mathrm{Sel}(E/F_\infty), \mathbb{Q}_p/\mathbb{Z}_p)$ compact Λ-module

There is a homomorphism with finite kernel and cokernel:

$$X(E/F_\infty) \to \Lambda^{\oplus r} \oplus \bigoplus_i \Lambda \big/ f_i(T)\Lambda \oplus \bigoplus_j \Lambda \big/ p^{m_j}\Lambda$$

where $f_i(T) \in \mathbb{Z}_p[T]$ are distinguished polynomials.

$\lambda(X(E/F_\infty)) := \sum_i \deg f_i$

$\mu(X(E/F_\infty)) := \sum_j m_j$

$\theta(X(E/F_\infty)) := \max_j m_j$

We will mostly assume $\theta \leq 1$.

$\pm$-Selmer groups

From now on: p-adic places have good but not necessarily ordinary reduction, i.e. $\forall v \mid p$: $\widetilde{E}$ is still an elliptic curve, but we allow $p \mid a_v$ $\rightsquigarrow$ supersingular places

Problem: local condition in Sel_{p^∞} is vacuous at supersingular places, and Sel_{p^∞} fails to be Λ-cotorsion!

Kobayashi (2000): new local conditions ($F = \mathbb{Q}$)

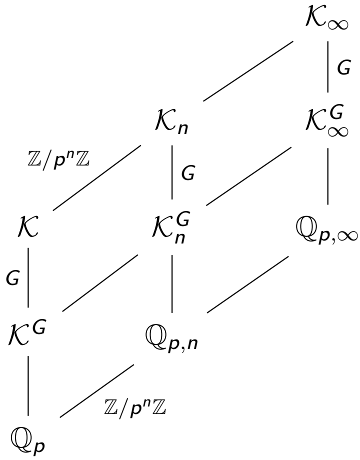
For $v \mid p$ supersingular and $n \geq 0$, define $\widehat{E}^{\pm}(F_{n,v}) :=$

$$\left\{ P \in E(F_{n,v}) : \forall 0 \leq m < n, m \text{ is } \frac{\text{even for } +}{\text{odd for } -} : \mathrm{Tr}_{F_{n,v}/F_{m+1,v}}(P) \in E(F_{m,v}) \right\}$$

Fix signs at supersingular places, i.e. choose $\vec{s} \in \{\pm\}^{\Sigma_p^{\mathrm{ss}}}$ (consistently)

$$\mathrm{Sel}^{\vec{s}}(E/F_n) := \ker \left(H^1(F_n, E[p^\infty]) \to \bigoplus_{\substack{v \mid p \\ v \in \Sigma_p^{\mathrm{ss}}}} \frac{H^1(F_{n,v}, E[p^\infty])}{\widehat{E}^{s_v}(F_{n,v}) \otimes \mathbb{Q}_p/\mathbb{Z}_p} \oplus (\cdots) \right)$$

$$\mathrm{Sel}^{\vec{s}}(E/F_\infty) := \varinjlim_n \mathrm{Sel}^{\vec{s}}(E/F_n), \quad X^{\vec{s}}(E/F_\infty) := \mathrm{Hom}\left(\mathrm{Sel}^{\vec{s}}(E/F_\infty), \mathbb{Q}_p/\mathbb{Z}_p\right)$$



$\mathcal{K}_\infty$
G
$\mathcal{K}_n$
$\mathcal{K}_\infty^G$
$\mathbb{Z}/p^n\mathbb{Z}$
G
$\mathcal{K}$
$\mathcal{K}_n^G$
$\mathbb{Q}_{p,\infty}$
G
$\mathcal{K}^G$
$\mathbb{Q}_{p,n}$
$\mathbb{Z}/p^n\mathbb{Z}$
$\mathbb{Q}_p$

$\mathcal{K}$ — localisation of F at a p-adic supersingular place

Assumptions: $\mathcal{K}/\mathbb{Q}_p$ finite unramified, $\mathcal{K}_\infty/\mathbb{Q}_p$ abelian
due to Kobayashi ($\mathcal{K} = \mathbb{Q}_p$) and B.D.Kim (generalisation)

Lubin–Tate theory $\Rightarrow$ $\widehat{E}(\mathcal{K}_n)$ is p-**torsion free** $\forall n$

Lubin–Tate theory + Honda theory $\rightsquigarrow$ system of $\pm$-norm coherent points

$0 \le i < [\mathcal{K} : \mathbb{Q}_p]$: $\quad c_{n,i} \in \widehat{E}(\mathcal{K}_n)$ generators, $\quad \mathrm{Tr}_{\mathcal{K}_n/\mathcal{K}_{n-1}}(c_{n,i}) = -c_{n-2,i}$

Consequences:

- $\widehat{E}(\mathcal{K}_n)$ is **cohomologically trivial** $\forall n$
- short exact sequence: $0 \to \widehat{E}(\mathcal{K}) \to \widehat{E}^+(\mathcal{K}) \oplus \widehat{E}^-(\mathcal{K}) \to \widehat{E}(\mathcal{K}_\infty) \to 0$
- **computing cohomology** of $\widehat{E}^\pm(\mathcal{K}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ and $\frac{H^1(\mathcal{K}_\infty, E[p^\infty])}{\widehat{E}^\pm(\mathcal{K}_\infty)\otimes\mathbb{Q}_p/\mathbb{Z}_p}$

Furthermore:

- existence of $\pm$-Coleman maps $\mathrm{Col}^\pm : \varprojlim H^1(\mathcal{K}_n, T_pE) \to \Lambda$
- control theorem for $\pm$-Selmer groups

w/o Assumptions: no system of norm coherent points! (that we know of)

Assumption 1: $p^2 - 1 \nmid e(\mathcal{K}/\mathbb{Q}_p)$.
$\Rightarrow \widehat{E}(\mathcal{K}_n)$ is p-torsion free $\forall n$ (similar proof as in unramified case)

Assumption 2: $\mathcal{K}/\mathbb{Q}_p$ weakly ramified and $\mathbb{Q}_{p,\infty} \cap \mathcal{K} = \mathbb{Q}_p$
Weakly ramified: $\mathrm{Gal}(\mathcal{K}/\mathbb{Q}_p) \supseteq G_0 \supseteq G_1 \supseteq G_2 = 1$
Assumption 2 $\Rightarrow \mathrm{Gal}(\mathcal{K}_\infty/\mathbb{Q}_p) \simeq \Gamma \times \mathrm{Gal}(\mathcal{K}/\mathbb{Q}_p)$
Assumption 2 $\Rightarrow \widehat{E}(\mathcal{K}_n)$ cohomologically trivial $\forall n$
proof uses Ellerbrock–Nickel ('18)

Goal: compute cohomology of $\widehat{E}^{\pm}(\mathcal{K}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ and $\frac{H^1(\mathcal{K}_\infty, E[p^\infty])}{\widehat{E}^{\pm}(\mathcal{K}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p}$

Avoiding Kobayashi's exact sequence via $\pm$-Iwasawa cohomology:

$H^{\pm}(\mathcal{K}_n, T_pE)$ — orthogonal complement of $\widehat{E}^{\pm}(\mathcal{K}_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ under the Tate pairing $H^1(\mathcal{K}_n, E[p^\infty]) \times H^1(\mathcal{K}_n, T_pE) \to \mathbb{Q}_p/\mathbb{Z}_p$

$H^{\pm}_{\mathrm{Iw}}(\mathcal{K}, T_pE) := \varprojlim_n H^{\pm}(\mathcal{K}_n, T_pE)$ — $\pm$-Iwasawa cohomology

Proposition (F.–Müller '24)

For $G \leq \mathrm{Gal}(\mathcal{K}/\mathbb{Q}_p)$ cyclic of order p: $H^{\pm}_{\mathrm{Iw}}(\mathcal{K}, T_pE)$ is free over $\Lambda(\Gamma \times G)$.

Proposition (Lim unramified, F.–Müller weakly ramified)

For all $G \leq \mathrm{Gal}(\mathcal{K}/\mathbb{Q}_p)$:

$$H^i\left(G, \frac{H^1(\mathcal{K}_\infty, E[p^\infty])}{\widehat{E}^\pm(\mathcal{K}_\infty)\otimes\mathbb{Q}_p/\mathbb{Z}_p}\right) = \begin{cases} \frac{H^1(\mathcal{K}_\infty^G, E[p^\infty])}{\widehat{E}^\pm(\mathcal{K}_\infty^G)\otimes\mathbb{Q}_p/\mathbb{Z}_p} & i = 0 \\ 0 & i > 0 \end{cases}$$

Sufficient to treat the case $G \simeq C_\ell$, ℓ prime. If $\ell \neq p$: obvious. Let $\ell = p$.
$H^\pm_{\mathrm{Iw}}(\mathcal{K}, T_pE)$ is free over $\Lambda(\Gamma \times G) \Rightarrow \forall i > 0$: $H^i(G, H^\pm_{\mathrm{Iw}}(\mathcal{K}, T_pE)) = 0$
Since $H^\pm_{\mathrm{Iw}}(\mathcal{K}, T_pE)$ is Tate dual to $\frac{H^1(\mathcal{K}_\infty, E[p^\infty])}{\widehat{E}^\pm(\mathcal{K}_\infty)\otimes\mathbb{Q}_p/\mathbb{Z}_p} \Rightarrow$ claim for $i > 0$

For $i = 0$: cyclicity of $G \Rightarrow \widehat{H}^0 = H^2 = 0$, so

$$\left(\frac{H^1(\mathcal{K}_\infty, E[p^\infty])}{\widehat{E}^\pm(\mathcal{K}_\infty)\otimes\mathbb{Q}_p/\mathbb{Z}_p}\right)_G = \mathrm{Tr}_G\left(\left(\frac{H^1(\mathcal{K}_\infty, E[p^\infty])}{\widehat{E}^\pm(\mathcal{K}_\infty)\otimes\mathbb{Q}_p/\mathbb{Z}_p}\right)\right)$$
$$\subseteq \frac{H^1(\mathcal{K}_\infty^G, E[p^\infty])}{\widehat{E}^\pm(\mathcal{K}_\infty^G)\otimes\mathbb{Q}_p/\mathbb{Z}_p} \qquad \square$$

In general, $X^{\vec{s}}(E/K_\infty)$ is not known to be $\Lambda = \Lambda(\Gamma)$-torsion
(even if there are no supersingular places, conjectured by Mazur '72)

If $X^{\vec{s}}(E/K_\infty)$ is Λ-torsion:

$\Leftrightarrow H^2(G_\Sigma(K_\infty), E[p^\infty]) = 0$ and there is a short exact sequence

$$\mathrm{Sel}^{\vec{s}}(E/K_\infty) \hookrightarrow H^1(G_\Sigma(K_\infty), E[p^\infty]) \twoheadrightarrow \bigoplus_{w \in \Sigma(K_\infty)} J_w(E/K_\infty)$$

where $G_\Sigma(K_\infty)$ is the Galois group of the maximal Σ-ramified extension

$\Rightarrow H^i(G, \mathrm{Sel}^{\vec{s}}(E/K_\infty))$ is finite for all $G \leq \mathrm{Gal}(K_\infty/F_\infty)$

$\Rightarrow X^{\vec{s}}(E/K_\infty)$ has no finite nonzero Λ-submodules

Proof buzzwords: Cassels–Poitou–Tate sequence, Greenberg resp. Hachimori–Matsuno at non-p-adic resp. ordinary primes, Hochschild–Serre spectral sequence.

Assume $X^{\vec{s}}(E/K_\infty)$ is Λ-torsion from now on.

Let $\Sigma' \subset \Sigma$ be a set of non-p-adic places.

$$\mathrm{Sel}^{\vec{s}}_{\Sigma'}(E/L_n) := \ker\left(H^1(G_\Sigma(L_n), E[p^\infty]) \to \bigoplus_{w \in (\Sigma - \Sigma')(L_n)} J_w(E/L_n) \right)$$

$$\mathrm{Sel}^{\vec{s}}_{\Sigma'}(E/L_\infty) := \varinjlim_n \mathrm{Sel}^{\vec{s}}_{\Sigma'}(E/L_n)$$

$$0 \to \mathrm{Sel}^{\vec{s}}(E/K_\infty) \to \mathrm{Sel}^{\vec{s}}_{\Sigma'}(E/K_\infty) \to \bigoplus_{w \in \Sigma'(K_\infty)} H^1(K_{\infty,w}, E[p^\infty]) \to 0$$

Inherits properties: same θ, $H^i(G, \mathrm{Sel}^{\vec{s}}_{\Sigma'}(E/K_\infty))$ finite, no finite submod.
Assume Σ' contains all places $v \nmid p$, $v \in \Sigma$ s.t. $p \mid f_v(K/F)$ inertia degree

Theorem (Lim unramified, F.–Müller weakly ramified)

If $X^{\vec{s}}(E/K_\infty)$ is Λ-torsion and $\theta(X^{\vec{s}}(E/K_\infty)) \leq 1$, then $X^{\vec{s}}_{\Sigma'}(E/K_\infty) \Big/ X^{\vec{s}}_{\Sigma'}(E/K_\infty)[p]$ is quasi-projective over $\mathbb{Z}_p[\mathrm{Gal}(K_\infty/F_\infty)]$.

Strictly quasi-projective: $0 \to$ finite $\to Y \to$ projective $\to$ finite $\to 0$
Quasi-projective: $0 \to$ sqp $\to$ sqp $\to Y \to 0$

Analogue of Riemann–Hurwitz formula, which relates Euler characteristics in a ramified covering of surfaces.

Theorem (Hachimori–Matsuno '98, Hachimori–Sharifi '05)

Let K/L be a finite Galois extension of p-power degree.
Suppose that $X(E/K_\infty)$ is Λ-torsion with $\theta(X(E/K_\infty)) \leq 1$.
Then $X(E/L_\infty)$ is also Λ-torsion, and we have:

$$\lambda(X(E/K_\infty)) = [K_\infty : L_\infty] \cdot \lambda(X(E/L_\infty)) + \sum_{w \in P_{\mathrm{spm}}} (e_w - 1) + 2 \sum_{w \in P_{\mathrm{good}}} (e_w - 1)$$

$$\mu(X(E/K_\infty)) = [K_\infty : L_\infty] \cdot \mu(X(E/L_\infty))$$

P_{spm}, $P_{\mathrm{good}} \subseteq \Sigma'$ — sets of places $w \nmid p$ of L_∞ above Σ, with E having split multiplicative resp. good reduction with $E(L_{\infty,w}) \neq 0$, and the inertia degree of the underlying primes in K/L being divisible by p

e_w — ramification index of w in K_∞/L_∞

$\pm$-Kida formulæ

$F/\mathbb{Q}$ — finite, $K/L/F$ — finite Galois p-extensions

Theorem (M.F.Lim '21)

There is a Kida formula $\mathrm{Sel}^{\vec{s}}(E/K_\infty) \leftrightsquigarrow \mathrm{Sel}^{\vec{s}}(E/L_\infty)$ *when:*

- *E/F has good reduction at p-adic places, at least one supersingular*
- *$\forall v \mid p$ supersingular place of F:*
  - $F_v = \mathbb{Q}_p$
  - *v is unramified in K/F*

Theorem (F.–Müller '24)

There is a Kida formula $\mathrm{Sel}^{\vec{s}}(E/K_\infty) \leftrightsquigarrow \mathrm{Sel}^{\vec{s}}(E/L_\infty)$ *when:*

- *E/F has good reduction at p-adic places, at least one supersingular*
- *$\forall v \mid p$ supersingular place of F:*
  - *p splits completely in $F/\mathbb{Q}$*
  - $p^2 - 1 \mid e_{K/F}(v)$
  - *$K_v \subseteq \mathcal{K}\mathbb{Q}_{p,\infty}$ for some $\mathcal{K}/\mathbb{Q}_p$ weakly ramified*

Kida formula: idea of proof (Hachimori–Sharifi)

$\Sigma' := \{v \in \Sigma : v \nmid p,\ p \mid f_v(K/F)\}$
Wlog $G = \mathrm{Gal}(K_\infty/L_\infty)$ cyclic of order p

$$\begin{array}{ccccc}
\mathrm{Sel}^{\vec{s}}_{\Sigma'}(E/L_\infty) & \hookrightarrow & H^1(G_\Sigma(L_\infty), E[p^\infty]) & \twoheadrightarrow & \bigoplus_{u \in (\Sigma - \Sigma')(L_\infty)} J_u(E/L_\infty) \\
\big\downarrow & & \sim\big\downarrow \mathrm{res} & & \twoheaddownarrow \\
\mathrm{Sel}^{\vec{s}}_{\Sigma'}(E/K_\infty)^G & \hookrightarrow & H^1(G_\Sigma(K_\infty), E[p^\infty])^G & \to & \bigoplus_{u \in (\Sigma - \Sigma')(K_\infty)} J_u(E/K_\infty)^G \\
\twoheaddownarrow & & & & \\
\text{finite} & & & &
\end{array}$$

$$\left.\begin{array}{l}
\mu((X^{\vec{s}}_{\Sigma'}(E/K_\infty))^G) = \mu(X^{\vec{s}}_{\Sigma'}(E/L_\infty)) \\
\forall i > 0 : \# H^i(G, \mathrm{Sel}^{\vec{s}}_{\Sigma'}(E/K_\infty)) < \infty
\end{array}\right\} \Rightarrow \mu(X^{\vec{s}}_{\Sigma'}(E/K_\infty)) = p\mu(X^{\vec{s}}_{\Sigma'}(E/L_\infty))$$

Also: $\lambda(X^{\vec{s}}_{\Sigma'}(E/K_\infty)) = p \cdot \lambda(X^{\vec{s}}_{\Sigma'}(E/L_\infty))$
Putting back the primes in Σ': corank computations due to Greenberg and Hachimori–Matsuno □

$\mathcal{Q}(\mathcal{G})$ — total ring of quotients of $\Lambda(\mathcal{G})$, i.e. localise at all non-zero-divisors
Y — finitely generated $\Lambda(\mathcal{G})$-module, torsion over $\Lambda = \Lambda(\Gamma)$, with a free $\Lambda(\mathcal{G})$-resolution of length 1: $0 \to \Lambda(\mathcal{G})^m \xrightarrow{\alpha} \Lambda(\mathcal{G})^m \to Y \to 0$
$[Y]$ — class of Y in $K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$: $[Y] := [(\Lambda(\mathcal{G})^m, \alpha, \Lambda(\mathcal{G})^m)]$
ξ_Y — characteristic element: $\xi_Y \in K_1(\mathcal{Q}(\mathcal{G}))$ s.t. $\partial(\xi_Y) = [Y]$, e.g. $[\alpha]$
Nota bene: ∂ is surjective (Witte)

Theorem (Greenberg, '11)

Y — fin.gen'd $\Lambda(\mathcal{G})$-mod., torsion/$\Lambda = \Lambda(\Gamma)$, no nonzero finite Λ-submod.

Then: Y admits a free resolution of length 1 of $\Lambda(\mathcal{G})$-modules if

$$\forall \mathcal{H} \leq \mathcal{G} : \quad H^1(\mathcal{H}, Y^\vee) = H^2(\mathcal{H}, Y^\vee) = 0$$

Greenberg's criterion is applicable to $X^{\vec{s}}_{\Sigma'}(E/K_\infty)$ if

- Σ' contains all places $v \nmid p$, $v \in \Sigma$ s.t. $p \mid f_v(K/F)$
- $X^{\vec{s}}(E/K_\infty)$ is Λ-torsion
- every ordinary p-adic place $v \in \Sigma_p^{\mathrm{ord}}$ is
  - either non-anomalous: if $w \mid v$ for w a place of K, then $p \nmid \#\tilde{E}(\overline{K_w})$
  - or ramifies tamely in K/F

Greenberg's criterion $\Rightarrow X^{\vec{s}}_{\Sigma'}(E/K_\infty)$ has a free resolution of length 1

Theorem (F, '23)

Y — fin.gen'd $\Lambda(\mathcal{G})$-module with a free resolution of length 1

Then: $\forall \mathfrak{M} \subset \mathcal{Q}(\mathcal{G})$ maximal orders: $\xi_Y \in \operatorname{im}\left(\mathfrak{M} \cap \mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G}))\right)$

$\Rightarrow$ integrality for $\xi_{X^{\vec{s}}_{\Sigma'}(E/K_\infty)}$

Corollary (F.–Müller '24)

Suppose that $X^{\vec{s}}(E/K_\infty)$ is Λ-torsion, and that every ordinary p-adic place $v \in \Sigma_p^{\mathrm{ord}}$ is either non-anomalous or tamely ramified.

Further suppose that $\Sigma' \cap P_{\mathrm{spm}} = \Sigma' \cap P_{\mathrm{good}} = \emptyset$.

Then: $\forall \mathfrak{M} \subset \mathcal{Q}(\mathcal{G})$ maximal: $\xi_{X^{\vec{s}}(E/K_\infty)} \in \operatorname{im}\left(\mathfrak{M} \cap \mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G}))\right)$

Recall: $\mathrm{Sel}^{\vec{s}}(E/K_\infty) \hookrightarrow \mathrm{Sel}^{\vec{s}}_{\Sigma'}(E/K_\infty) \twoheadrightarrow \bigoplus_{w \in \Sigma'(K_\infty)} H^1(K_{\infty,w}, E[p^\infty])$

$\Sigma' \cap P_{\mathrm{spm}} = \Sigma' \cap P_{\mathrm{good}} = \emptyset \Rightarrow H^1(K_{\infty,w}, E[p^\infty])$ finite $\forall w \in \Sigma'(K_\infty)$

$\Rightarrow \left[X^{\vec{s}}(E/K_\infty)\right] = \left[X^{\vec{s}}_{\Sigma'}(E/K_\infty)\right] \in K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$

$\Rightarrow \xi_{X^{\vec{s}}(E/K_\infty)} = \xi_{X^{\vec{s}}_{\Sigma'}(E/K_\infty)}$ $\square$