# Divided Powers I

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PD := puissances divisées = divided powers

## Definition (PD structure)

$(A, I, \gamma)$ PD ring: $A$ ring, $I \subseteq A$ ideal, $\gamma = (\gamma_i : I \to A)_{i \geqslant 0}$ satisfying

1. $\gamma_0(x) = 1$, $\gamma_1(x) = x$, $\gamma_i(x) \in I$ $\qquad \forall x \in I$, $\forall i \geqslant 1$
2. $\gamma_n(x + y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y)$ $\qquad \forall x, y \in I$
3. $\gamma_i(\lambda x) = \lambda^i \gamma_i(x)$ $\qquad \forall x \in I, \forall \lambda \in A$
4. $\gamma_i(x)\gamma_j(x) = \binom{i+j}{i}\gamma_{i+j}(x)$ $\qquad \forall x \in I$
5. $\gamma_i(\gamma_j(x)) = \frac{(ij)!}{i!(j!)^i}\gamma_{ij}(x)$ $\qquad \forall x \in I$

$n!\gamma_n(x) = x^n$ follows from $\gamma_1(x) = x$ and using 4. inductively:

$$x \cdot x^n = \gamma_1(x) \cdot n!\gamma_n(x) \overset{4.}{=} n! \cdot \binom{n+1}{1}\gamma_{n+1}(x)$$

$\Rightarrow$ in char 0, there is at most one PD structure (not true in positive char)

## Examples

- $A$ any ring, $I = (0)$, $\gamma_n = 0$ $(n \geqslant 1)$
- $A$ a $\mathbb{Q}$-algebra, $I$ any ideal, $\gamma_n(x) = x^n/n!$

# Example: discrete valuation rings

$A$ DVR of characteristic 0

$p$ the residue field of characteristic

$\pi$ a uniformiser

$e$ the absolute ramification index, i.e. $p = u\pi^e$ for some $u \in A^\times$

## Claim

$(A, (\pi))$ has a PD structure $\Leftrightarrow$ $e \leqslant p - 1$

## Proof

characteristic 0 $\Rightarrow \gamma_n(x) = \frac{x^n}{n!}$

Does this lie in $(\pi)$? Suffices to check for $x = \pi$:

$$\begin{aligned} \operatorname{ord}_\pi\left(\frac{\pi^n}{n!}\right) &= n - \operatorname{ord}_\pi(n!) = n - e \operatorname{ord}_p(n!) \\ &= \frac{p-1-e}{p-1} n + e \frac{\sum_i a_i}{p-1} \qquad \text{where } n = \sum_i a_i p^i \end{aligned}$$

$\gamma_n(\pi) \in (\pi)$ $\Leftrightarrow$ $\operatorname{ord}_\pi(\gamma_n(\pi)) \geqslant 1$ $\Leftrightarrow$ $p - 1 - e \geqslant 0$ $\square$

# More definitions

## Definition (PD morphism)

$f : (A, I, \gamma) \to (B, J, \delta)$ *PD morphism*: $f : A \to B$ ring homomorphism, $f(I) \subseteq J$, and the following squares commute for all $n \geqslant 1$:

$$\begin{array}{ccc} I & \xrightarrow{f} & J \\ \gamma_n \downarrow & \circlearrowleft & \downarrow \delta_n \\ I & \xrightarrow{f} & J \end{array}$$

## Definition (sub-PD ideal)

$(A, I, \gamma)$ PD ring, $J \subseteq I$ an ideal. $J$ is a *sub-PD ideal* if $\gamma_n(J) \subseteq J \ \forall n \geqslant 1$.

## Lemma (PD structure on a quotient ring)

*$(A, I, \gamma)$ PD ring, $J \subseteq A$ ideal, $\bar{I} := I(A/J)$. Then*

$$\exists! \bar{\gamma} \text{ PD structure on } \bar{I} \text{ s.t. } (A, I, \gamma) \to (A/J, \bar{I}, \bar{\gamma}) \text{ is a PD morphism}$$

$$\Leftrightarrow \; I \cap J \subseteq I \text{ is a sub-PD ideal.}$$

## Proof

Uniqueness: $\bar{\gamma}_n(\bar{x}) = \overline{\gamma_n(x)}$ for $x \in I$ a lift of $\bar{x} \in \bar{I}$

Existence: need to check well-definedness. Let $y \in I \cap J$.

$$\gamma_n(x+y) = \sum_{k+\ell=n} \gamma_k(x)\gamma_\ell(y) = \gamma_n(x) + \sum_{\ell=1}^{n} \gamma_{n-\ell}(x) \underbrace{\gamma_\ell(y)}_{\in I \cap J}$$

$\Rightarrow \gamma_n(x) \equiv \gamma_n(x+y) \pmod{I \cap J}$. PD axioms obviously hold.

Conversely: $x \in I \cap J \Rightarrow 0 = \bar{\gamma}_n(0) = \bar{\gamma}_n(\bar{x}) = \overline{\gamma_n(x)} \Rightarrow \gamma_n(x) \in I \cap J$ □

### Lemma (being a sub-PD ideal can be checked on a generating set)

*$(A, I, \gamma)$ PD ring, $S \subseteq I$ a subset, $J := (S) \subseteq I$ the generated ideal. Then*

$$J \text{ sub-PD ideal (i.e. } \gamma_n(J) \subseteq J) \;\Leftrightarrow\; \forall n \geqslant 1 : \gamma_n(S) \subseteq J.$$

### Proof

"$\Rightarrow$" obvious by definition.

"$\Leftarrow$" $J' := \{x \in J \,|\, \forall n \geqslant 1 : \gamma_n(x) \in J\}$. Suffices to show: $J' \subseteq J$ ideal.

$$\gamma_n(x+y) = \sum_{k+\ell} \underbrace{\gamma_k(x)}_{\in J} \underbrace{\gamma_\ell(y)}_{\in J} \qquad \forall x, y \in J'$$

$$\gamma_n(\lambda x) = \lambda^n \underbrace{\gamma_n(x)}_{\in J} \qquad \forall x \in J', \forall \lambda \in A$$

□

### Definition (set of PD generators)

$(A, I, \gamma)$ PD ring. $S \subseteq I$ is a *set of PD generators of $I$* if $I$ is the smallest sub-PD ideal of $I$ containing $S$. Equivalently: $I = (\gamma_n(s) \mid s \in S, n \geqslant 1)$.

### Proposition (products of PD ideals)

*Let $(I, \gamma)$ and $(J, \delta)$ be PD ideals of $A$. Then $IJ$ is a sub-PD ideal of both $I$ and $J$, and $\gamma|_{IJ} = \delta|_{IJ}$.*

### Proof

Lemma $\Rightarrow$ sufficient to check on generators $xy \in IJ$, $x \in I$, $y \in J$.

$$\gamma_n(xy) = y^n \gamma_n(x) = n! \underbrace{\delta_n(y)}_{\in J} \underbrace{\gamma_n(x)}_{\in I} = x^n \delta_n(y) = \delta_n(xy)$$

□

### Corollary (Powers of PD ideals)

*$(I, \gamma)$ PD ideal $\Rightarrow I^n \subseteq I$ sub-PD ideal for all $n \geqslant 1$.*

### Proof

Induction. □

### Proposition (PD structure on sum)

*$(I, \gamma)$ and $(J, \delta)$ PD ideals of $A$, $I \cap J$ sub-PD ideal of both, $\gamma|_{I \cap J} = \delta|_{I \cap J}$.*

*Then $I + J$ has a unique PD structure $\varepsilon$ s.t. $I$ and $J$ become sub-PD ideals.*

### Proof (naïve way)

Let $\varepsilon_n(x + y) := \sum_{i+j=n} \gamma_i(x)\delta_j(y)$, then do a *lot* of calculations. $\square$

## Definition

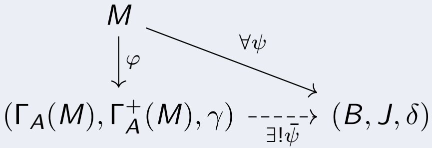
$$\Gamma_A : A\text{-Mod} \to A\text{-Alg},\, M \mapsto \frac{A\left[(x,n) \,\middle|\, x \in M, n \geqslant 0\right]}{\begin{array}{c}(x,0)-1, (\lambda x, n) - \lambda^n (x,n), \\ (x,n)(x,m) - \binom{n+m}{n}(x, n+m), \\ (x+y, n) - \sum_{i+j=n} (x,i)(y,j)\end{array}}$$

$x^{[n]} :=$ the image of $(x,n)$; $\varphi : M \to \Gamma_A(M)$, $x \mapsto x^{[1]}$

$\Gamma_A(M)$ becomes a graded algebra by letting $\deg(x,n) := n$. $\quad \Gamma_A^1(M) \simeq M$

## Theorem

*$M$ an $A$-module. Then $(\Gamma_A(M), \Gamma_A^+(M))$ admits a unique PD structure $\gamma$ possessing the following universal property:*




*for every $A$-PD algebra $(B, J, \delta)$ and every $A$-linear map $\psi : M \to J$ $\exists! \bar{\psi}$ PD morphism making the diagram commute.*

Properties of Roby's algebra $\Gamma_A(M)$:

- $\Gamma_A^n(M) = \left\langle x^{[q]} = x_1^{[q_1]} \cdots x_k^{[q_k]} \,\middle|\, \sum_{i=1}^k q_i = n, x_i \in M \right\rangle_A$
- $\Gamma_{A'}(M \otimes_A A') \simeq A' \otimes_A \Gamma_A(M)$ for any $A$-algebra $A'$
- $\Gamma_A\left(\varinjlim M_i\right) \simeq \varinjlim \Gamma_A(M_i)$
- $\Gamma_A(M \oplus N) \simeq \Gamma_A(M) \otimes_A \Gamma_A(N)$

## Definition

For an $A$-algebra $R$, the $A$-module of *power series of exponential type* is:

$$\exp(R) := \{f \in R[\![T]\!] \mid f(0) = 1, f(T_1 + T_2) = f(T_1)f(T_2)\}$$

## Proposition ($\Gamma_A$–exp adjunction)

$$\Gamma_A : A\text{-Mod} \to A\text{-Alg} \dashv \exp : A\text{-Alg} \to A\text{-Mod}$$

$$\mathrm{Hom}_{A\text{-Alg}}(\Gamma_A(M), R) \simeq \mathrm{Hom}_{A\text{-Mod}}(M, \exp(R))$$

$$\alpha \mapsto \beta : x \mapsto \sum_{n=0}^{\infty} \alpha\left(x^{[n]}\right) T^n$$

## Theorem

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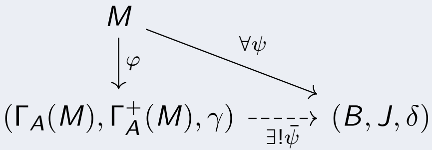


*for every $A$-PD algebra $(B, J, \delta)$ and every $A$-linear map $\psi : M \to J$ $\exists!\bar{\psi}$ PD morphism making the diagram commute.*

## Proof of the universal property

$\tilde{\psi} : M \to \exp(B), x \mapsto \sum_{n=0}^{\infty} \delta_i(\psi(x))T^i$

$\Gamma_A \dashv \exp \Rightarrow \exists!\bar{\psi} : \Gamma_A(M) \to B, x^{[n]} \mapsto \delta_n(\psi(x))$ PD morphism?

$$\bar{\psi}\left(\gamma_i\left(x^{[n]}\right)\right) = \bar{\psi}\left(\frac{(ni)!}{n!(i!)^n}x^{[ni]}\right) = \frac{(ni)!}{n!(i!)^n}\bar{\psi}\left(x^{[ni]}\right)$$
$$= \frac{(ni)!}{n!(i!)^n}\delta_{ni}\left(\psi(x)\right) = \delta_i\left(\delta_n(\psi(x))\right) = \delta_i\left(\bar{\psi}\left(x^{[n]}\right)\right)$$

Uniqueness: $\varphi(M) = \Gamma_A^1(M)$ generates $\Gamma_A(M)$. □

## Proposition (PD structure on sum)

*$(I, \gamma)$ and $(J, \delta)$ PD ideals of $A$, $I \cap J$ sub-PD ideal of both, $\gamma|_{I\cap J} = \delta|_{I\cap J}$.*

*Then $I + J$ has a unique PD structure $\varepsilon$ s.t. $I$ and $J$ become sub-PD ideals.*

## Proof (clever way)

$g : I \to \exp(A), x \mapsto \sum_{n=0}^{\infty} \gamma_n(x) T^n$ $\qquad$ $d : J \to \exp(A), y \mapsto \sum \delta_n(y) T^n$

$0 \to I \cap J \to I \oplus J \to I + J \to 0$, universal property of $I \oplus J$:

$\exists! e : I + J \to \exp(A)$ inducing $g, d$ with Taylor expansion $z \mapsto \sum \varepsilon_n(z) T^n$

Check PD axioms: e.g. $\varepsilon_k(\varepsilon_\ell(z)) = \frac{(k\ell)!}{k!(\ell!)^k} \varepsilon_{k\ell}(z)$. Fix $z = x + y \in I + J$.

$$\begin{array}{cc} \Gamma_A(\langle X \rangle_A) \to (A, I, \gamma) & \Gamma_A(\langle Y \rangle_A) \to (A, J, \delta) \\ X \mapsto x & Y \mapsto y \end{array}$$

$$\begin{array}{ccc} \Gamma_A\big(\langle X \rangle_A \oplus \langle Y \rangle_A\big) \simeq \Gamma_A(\langle X \rangle_A) \otimes_A \Gamma_A(\langle Y \rangle_A) & \to & A \quad A\text{-algebra morphism} \\ X + Y & \longmapsto & x + y \end{array}$$

$$\left((X+Y)^{[k]}\right)^{[\ell]} = \frac{(k\ell)!}{k!(\ell!)^k}(X+Y)^{[k\ell]} \Rightarrow \varepsilon_k(\varepsilon_\ell(x+y)) = \frac{(k\ell)!}{k!(\ell!)^k}\varepsilon_{k\ell}(x+y) \qquad \square$$

## Definition (extension of PD structure)

$(A, I, \gamma)$ PD ring. Then $\gamma$ extends to an $A$-algebra $B$ if there is a PD structure $\bar{\gamma}$ such that $(A, I, \gamma) \to (B, IB, \bar{\gamma})$ is a PD morphism.

$\bar{\gamma}$ is unique (if it exists): $\bar{\gamma}_n(bx) = b^n \bar{\gamma}_n(x) = b^n \gamma_n(x)\ \forall x \in I, \forall b \in B$

## Proposition

*If $I = (t)$ is principal then $\gamma$ extends.*

## Proof

$\gamma_n(tb) := b^n \gamma_n(t)$, need to check well-definedness. If $tb_1 = tb_2$ then

$$\bar{\gamma}_n(tb_1) - \bar{\gamma}_n(tb_2) = \underbrace{(b_1^n - b_2^n)}_{(b_1 - b_2)|} \underbrace{\gamma_n(t)}_{t|} = 0$$

Verifying PD axioms is easy. □

### Definition (extension of PD structure)

$(A, I, \gamma)$ PD ring. Then $\gamma$ extends to an $A$-algebra $B$ if there is a PD structure $\bar{\gamma}$ such that $(A, I, \gamma) \to (B, IB, \bar{\gamma})$ is a PD morphism.

### Definition (sub-PD algebra)

$(A, I, \gamma)$ PD algebra, $C \subseteq A$ a subalgebra. $C$ is a *sub-PD algebra* if $\gamma_n(I \cap C) \subseteq I \cap C \; \forall n \geqslant 1$.

## Proposition-Definition (compatibility)

$(A, I, \gamma)$ PD ring, $(B, J, \delta)$ $A$-PD-algebra. Then $\gamma$ and $\delta$ are *compatible* if the following equivalent conditions hold:

1. $\gamma$ has an extension $\bar{\gamma}$ to $B$ and $\bar{\gamma}|_{IB \cap J} = \delta|_{IB \cap J}$
2. $IB + J$ has a unique PD structure $\bar{\delta}$ with $(A, I, \gamma) \to (B, IB + J, \bar{\delta})$ and $(B, J, \delta) \to (B, IB + J, \bar{\delta})$ PD morphisms
3. $\exists K' \supseteq IB + J$ ideal of $B$ with PD structure $\delta'$ and $(A, I, \gamma) \xrightarrow{\varphi} (B, K', \delta')$ and $(B, J, \delta) \to (B, K', \delta')$ PD morphisms

## Proof

$1 \Rightarrow 2$: this is the lemma about PD structure on the sum of ideals

$2 \Rightarrow 3$: $K' := IB + J$

$3 \Rightarrow 1$: Claim: $IB$ is a sub-PD ideal of $K'$.
Suffices to check this on generators $x \in I$: $\varphi(\gamma_n(x)) = \delta'_n(\varphi(x))$.
Define $\bar{\gamma} := \delta'|_{IB}$.
$\bar{\gamma}|_{IB \cap J} = \delta|_{IB \cap J}$ because $(B, J, \delta) \to (B, K', \delta')$ PD morphism. □

# References

Pierre Berthelot, Arthur Ogus
*Notes on Crystalline Cohomology*.
Princeton University Press and University of Tokyo Press, 1978.

The Stacks project authors
*The Stacks project*, Section 09PE.
https://stacks.math.columbia.edu, 2020.