Affinoid Algebras and Affinoid Spaces

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$\rightsquigarrow$ think of $T_n/\mathfrak{a}$ as the algebra of functions on $V(\mathfrak{a})$

Definition (Definition 3.1/1)

Affinoid K-algebra: homomorphic image of a Tate algebra, that is,

$$\exists \alpha : T_n \twoheadrightarrow A \text{ epimorphism of } K\text{-algebras}$$

Proposition (Proposition 3.1/3)

Affinoid K-algebras

1. *are Noetherian*
2. *are Jacobson, i.e. prime ideals are intersections of maximal ideals*
3. *satisfy Noether normalisation, i.e.* $\exists T_d \hookrightarrow A$ *finite monomorphism*

These are direct consequences of the corresponding results for T_n. [2.2/11+14+16]

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$\alpha : T_n \twoheadrightarrow A$ affinoid K-algebra

Definition (Residue norm)

$$|\alpha(f)|_\alpha := \inf\{|f - a| : a \in \operatorname{Ker}\alpha\}$$

In words: the distance of a preimage in the Tate algebra from $\operatorname{Ker}\alpha$.

Proposition (Proposition 3.1/5)

- *$|\cdot|_\alpha$ is a K-algebra norm (ring norm + K-vector space norm),*
- *α is continuous and open.*
- *$|\cdot|_\alpha$ induces the quotient topology, and this topology is complete.*

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Definition (Value of a function at a prime)

$\operatorname{Sp} A := \{\text{maximal ideals}\}$ (without a topology for now...)

For $\mathfrak{m} \in \operatorname{Sp} A$, let $f(\mathfrak{m})$ be the image of f under $A \twoheadrightarrow A/\mathfrak{m}$.

$A/\mathfrak{m}$ is a finite field extension of K [2.2/12], so $|f(\mathfrak{m})|$ makes sense.

Definition (Supremum 'norm')

$$|f|_{\sup} := \sup\{|f(\mathfrak{m})| : \mathfrak{m} \in \operatorname{Sp} A\}$$

If $A = T_n/\mathfrak{a}$, this is just the supremum of f on $V(\mathfrak{a}) \subset \mathbb{B}^n(\overline{K})$.

$|\cdot|_{\sup}$ is a semi-norm, i.e. $f = 0 \not\Leftarrow |f|_{\sup} = 0 \;\Leftrightarrow\; f$ nilpotent ($f^k = 0$)

"$\Leftarrow$": $0 = |f^k|_{\sup} = |f|_{\sup}^k$ because $|\cdot|_{\sup}$ is power multiplicative

"$\Rightarrow$": $|f|_{\sup} = 0 \Rightarrow f \in \operatorname{Jac}(A) = \operatorname{nil}(A)$ because A is Jacobson. □

Proposition (Proposition 3.1/7: the sup norm is contractive)

$\varphi : B \to A$ affinoid K-algebra morphism. Then $\forall b \in B$: $|\varphi(b)|_{\sup} \leqslant |b|_{\sup}$.

$|\varphi(b)(\mathfrak{m})| = |b(\varphi^{-1}(\mathfrak{m}))|$ □

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On the Tate algebra T_n, $|\cdot|_{\sup} = |\cdot|$.

Lemma (Corollary 2.2/13)

$$\mathbb{B}^n(\overline{K}) \twoheadrightarrow \operatorname{Sp} T_n$$
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$$|f|_{\sup} = \sup\{|f(\mathfrak{m})| : \mathfrak{m} \in \operatorname{Sp} T_n\} \underset{\text{Lemma}}{=} \sup\{|f(x)| : x \in \mathbb{B}^n(\overline{K})\} \underset{\substack{\text{Maximum}\\\text{Principle}}}{=} |f| \quad \square$$

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Proposition (Proposition 3.1/9)

$|\cdot|_{\text{sup}} \leqslant |\cdot|_\alpha$ for an affinoid K-algebra $\alpha : T_n \twoheadrightarrow A$.

Let $f \in A$, $g \in T_n$ a preimage of f such that $|f|_\alpha = |g|$. Let $\mathfrak{m} \in \operatorname{Sp} A$.

$$|f(\mathfrak{m})| \underset{\text{def}}{=} \left|g\left(\alpha^{-1}(\mathfrak{m})\right)\right| \underset{\text{def of } |\cdot|_{\text{sup}}}{\leqslant} |g|_{\text{sup}} \underset{3.1/8}{=} |g| = |f|_\alpha$$

Now let $\mathfrak{m}$ run through $\operatorname{Sp} A$, and use $|f|_{\text{sup}} \overset{\text{def}}{=} \sup\{|f(\mathfrak{m})| : \mathfrak{m} \in \operatorname{Sp} A\}$. □

Further results:

- A Sp-variant of the maximum principle holds for affinoid algebras.
- $|f|_{\text{sup}} \leqslant 1 \;\Leftrightarrow\; f$ is power bounded wrt residue norm topology
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Proof via associating a unique monic polynomial with f whose coefficients describe $|f|_{\text{sup}}$.

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$|\cdot|_{\sup} \leqslant |\cdot|_\alpha$ *for an affinoid K-algebra $\alpha : T_n \twoheadrightarrow A$.*

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Lemma (Lemma 3.1/19)

Let $f_1, \dots, f_n \in A$.

(i) *If $\varphi : K\langle\zeta_1, \dots, \zeta_n\rangle \to A$ is a K-morphism such that $\varphi(\zeta_i) = f_i$ then $|f_i|_{\sup} \leqslant 1$ for all i.*

(ii) *Conversely: if $|f_i|_{\sup} \leqslant 1$ for all i then $\exists! \, \varphi : K\langle\zeta_1, \dots, \zeta_n\rangle \to A$ K-morphism such that $\varphi(\zeta_i) = f_i$.*

Moreover, φ is continuous wrt the Gauß norm $|\cdot|$ on T_n and any residue norm $|\cdot|_\alpha$ on A.

(i) is just contractivity: $|f_i|_{\sup} \leqslant |\zeta_i|_{\sup} = |\zeta_i| = 1$; (ii) is more involved.

Proposition (Proposition 3.1/20)

Any morphism $B \to A$ of affinoid K-algebras is continuous wrt any residue norms. Hence all residue norms are equivalent.

$T_n \twoheadrightarrow B \to A$ is continuous by Lemma. □

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Moreover, φ *is continuous wrt the Gauß norm* $|\cdot|$ *on* T_n *and any residue norm* $|\cdot|_\alpha$ *on* A.

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Remarks:

- The sup norm is equivalent to residue norms whenever A is reduced.
- Proposition 20 can be shown using OMT & CGT.

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Theorem (Open Mapping Theorem)

X, Y Banach spaces, $\mathcal{A} : X \twoheadrightarrow Y$ surjective continuous linear operator. Then $\mathcal{A}$ is open.

Theorem (Closed Graph Theorem)

A linear operator between Banach spaces is continuous iff its graph is closed.

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$\operatorname{Sp} A := \{\text{maximal ideals}\}$

Why not all primes? Bosch: *"[W]e will introduce a certain process of localization for affinoid K-algebras [...]. Similarly as in algebraic geometry, this localization process is used in order to endow affinoid K-spaces with the structure of a ringed space. As only maximal ideals behave well with respect to localization in this sense, we must restrict ourselves to spectra of maximal ideals."* $\rightsquigarrow$ Talk 7

If $\mathfrak{p} \in \operatorname{Spec} A$ then $K_\mathfrak{p} = \operatorname{Frac}(A/\mathfrak{p})$ may be an infinite field extension of K.

- $K_\mathfrak{p}$ is not an aff'd K-algebra: it contradicts Noether normalisation.
- We have no absolute value on $K_\mathfrak{p}$ extending that on K and making $K_\mathfrak{p}$ complete.

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Definition (Vanishing set)

$V(\mathfrak{a}) := \{x \in \operatorname{Sp}(A) : \forall f \in \mathfrak{a} : f(x) = 0\}$

We have the usual things from algebraic geometry:

- Zariski topology
- vanishing ideal $\operatorname{id}(Y) := \{f \in A : \forall y \in Y, f(y) = 0\}$
- Hilbert's Nullstellensatz: $\operatorname{id}(V(\mathfrak{a})) = \operatorname{rad} \mathfrak{a}$
- $V(\cdot)$ and $\operatorname{id}(\cdot)$ are mutually inverse bijections

Definition (Morphisms of affinoid K-spaces)

$\sigma : B \to A$ induces ${}^a\sigma : \operatorname{Sp} A \to \operatorname{Sp} B$, $\mathfrak{m} \mapsto \sigma^{-1}(\mathfrak{m})$

This makes sense: $K \hookrightarrow B/\sigma^{-1}(\mathfrak{m}) \overset{\sigma}{\hookrightarrow} A/\mathfrak{m}$ because $\operatorname{Ker} \sigma \subseteq \sigma^{-1}(\mathfrak{m})$.

Alternate notation: $\varphi : \operatorname{Sp} A \to \operatorname{Sp} B$ induced by $\varphi^* : B \to A$.

$$\begin{array}{ccc} B & \xrightarrow{\varphi^*} & A \\ \downarrow & & \downarrow \\ B/\mathfrak{m}_{\varphi(x)} & \longrightarrow & A/\mathfrak{m}_x \end{array}$$

$$\mathfrak{m}_x := \{f \in A : f(x) = 0\}$$
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The category of affinoid K-spaces is the opposite of the category of affinoid K-algebras.

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Definition (Definition 3.3/1)

Let $X = \operatorname{Sp} A$, $f \in A$, $\varepsilon > 0$. Then $X(f, \varepsilon) := \{x \in X : |f(x)| \leqslant \varepsilon\}$.

These generate the so-called canonical topology of X.

Notation: $X(f) := X(f, 1)$, $X(f_1, \ldots, f_r) := X(f_1) \cap \cdots \cap X(f_r)$

Proposition (Proposition 3.3/2)

The canonical topology is generated by $\{X(f) : f \in A\}$. Hence a subset is open iff it is a union of sets $X(f_1, \ldots, f_r)$.

Proof: routine point set topology + valuation theory.

Proposition (Proposition 3.3/5)

The sets $X(f_1, \ldots, f_r)$ for $f_1, \ldots, f_r \in \mathfrak{m}_x \in \operatorname{Sp} A$ and $r \geqslant 1$ form a basis of neighbourhoods of $x \in X = \operatorname{Sp} A$.

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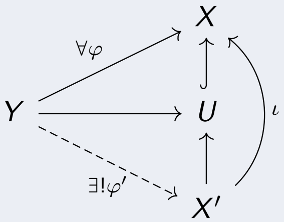
Definition (Definition 3.3/9, Affinoid subdomains)

X affinoid K-space, $U \subseteq X$ is an affinoid subdomain if $\exists \iota : X' \to X$ morphism of affinoid K-spaces such that $\iota(X') \subseteq U$ and ι is universal:



It actually follows that $\iota : X' \to U$ is a bijection, and we will suppress ι from now on. Thus an affinoid subdomain has a unique affinoid K-space structure.

More reasons for considering $\operatorname{Sp} A$ instead of $\operatorname{Spec} A$:

Let $U \hookrightarrow X$ be an affinoid subdomain, $A \to A'$ the corresponding morphism of affinoid K-algebras.

If $\mathfrak{p} \in \operatorname{Spec} A$ then $\mathfrak{p}A'$ may not even be prime. (A point in X may correspond to two distinct points in U.)

Example 3.3/20:

$$X = \operatorname{Sp} T_1 = \operatorname{Sp} K\langle\zeta\rangle \qquad c \in K, |c| < 1$$

$$U = \{x \in X : |\zeta(x)(\zeta(x)-1)| \leqslant |c|\} = \{|\zeta(x)| \leqslant |c|\} \sqcup \{|\zeta(x)-1| \leqslant |c|\}$$

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Definition (Definition 3.3/7, Special subdomains)

(i) $X(f_1,\dots,f_r) := \{x \in X : |f_i(x)| \leqslant 1\}$ Weierstraß domain

(ii) $X(f_1,\dots,f_r,g_1^{-1},\dots,g_s^{-1}) := \{x \in X : |f_i(x)| \leqslant 1, |g_j(x)| \geqslant 1\}$ Laurent domain

(iii) $X\left(\frac{f_1}{f_0},\dots,\frac{f_r}{f_0}\right) := \{x \in X : |f_i(x)| \leqslant |f_0(x)|\}$ rational domain, where $f_0,\dots,f_r \in A$ have no common zeros

On the 'no common zeros' condition:

- $f_0(x) \neq 0$ because that would mean common zeros
- The condition is necessary to ensure openness. Example: $X = \operatorname{Sp}\mathbb{Q}_p\langle\zeta_1\rangle$; $X\left(x \in X : |\zeta_1(x)| \leqslant |p\zeta_1(x)|\right) = \{(\zeta_1)\}$ is not open.

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These domains are open in the canonical topology. The Weierstraß domains even form a basis.

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Theorem (Theorem 3.3/20, Gerritzen–Grauert)

Any affinoid subdomain is a finite union of rational subdomains. (Hence all aff'd subdomains are open. But: not every finite union is aff'd.)

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In particular if $Y = X'$, $\varphi = \iota$, we get $\iota(X') \subseteq X(f)$:

$$|\iota^*(f_i)|_{\text{sup}} = |\overline{\zeta_i}|_{\text{sup}} \overset{\text{contractivity}}{\leqslant} |\zeta_i|_{\text{sup}} = |\zeta_i| = 1$$

$\overline{\zeta_i} \in A\langle f\rangle$ is the residue class of $\zeta_i \in A\langle\zeta\rangle$

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Universal property: wts $\forall \varphi^* : A \to B$ with $|\varphi^*(f_i)|_{\sup} \leqslant 1$ admits a unique factorisation through $\iota^* : A \to A\langle f\rangle$.

$\varphi^* : A \to B$ extends to $A\langle\zeta\rangle \to B$, $\zeta_i \mapsto \varphi^*(f_i)$, which has $\zeta_i - f_i$ in the kernel, thus descending to $A\langle f\rangle \to B$, so φ^* factors through ι^*.
Uniqueness: A has dense image in $A\langle f\rangle$.

Wts: $X(f, g^{-1}) = \{x \in X : |f_i(x)| \leqslant 1, |g_j(x)| \geqslant 1\}$ is an aff'd subdomain.

$$A\langle\zeta\rangle := A\langle\zeta_1, \ldots, \zeta_r\rangle := \left\{\sum_\nu a_\nu \zeta^\nu \in A[[\zeta]] : a_\nu \in A, \lim a_\nu = 0\right\}$$

$$A\langle f, g^{-1}\rangle := A\langle\zeta, \xi\rangle \Big/ (\zeta_i - f_i, 1 - g_j\xi_j : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s)$$

Take $\iota^* : A \to A\langle f, g^{-1}\rangle$ canonical; $\iota : X' := \operatorname{Sp} A\langle f, g^{-1}\rangle \to X = \operatorname{Sp} A$.

Let $\varphi : Y = \operatorname{Sp} B \to X$, $\varphi^* : A \to B$.

$$\begin{aligned} \varphi(Y) \subseteq X(f, g^{-1}) &\Leftrightarrow \forall y, \forall i : |\varphi^*(f_i(y))| \leqslant 1 \,\&\, \forall j : |\varphi^*(g_j(y))| \geqslant 1 \\ &\Leftrightarrow \forall i : |\varphi^*(f_i)|_{\text{sup}} \leqslant 1 \,\&\, \forall j : |\varphi^*(g_j)^{-1}|_{\text{sup}} \leqslant 1 \end{aligned}$$

Wts: $X(f, g^{-1}) = \{x \in X : |f_i(x)| \leqslant 1, |g_j(x)| \geqslant 1\}$ is an aff'd subdomain.

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Take $\iota^* : A \to A\langle f, g^{-1}\rangle$ canonical; $\iota : X' := \operatorname{Sp} A\langle f, g^{-1}\rangle \to X = \operatorname{Sp} A$.

Let $\varphi : Y = \operatorname{Sp} B \to X$, $\varphi^* : A \to B$.

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In particular if $Y = X'$, $\varphi = \iota$, we get $\iota(X') \subseteq X(f, g^{-1})$:

$$|\iota^*(f_i)|_{\text{sup}} \leqslant 1 \text{ as before}$$

$$\left|\iota^*(g_j)^{-1}\right|_{\text{sup}} = |\overline{\xi_j}|_{\text{sup}} \overset{\text{contractivity}}{\leqslant} |\xi_j|_{\text{sup}} = |\xi_j| = 1$$

$A\langle\zeta, \xi\rangle \twoheadrightarrow A\langle f, g^{-1}\rangle$, $\xi_j \mapsto \overline{\xi_j}$

Wts: $X(f, g^{-1}) = \{x \in X : |f_i(x)| \leqslant 1, |g_j(x)| \geqslant 1\}$ is an aff'd subdomain.

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Universal property: want to show $\forall \varphi^* : A \to B$ with $|\varphi^*(f_i)|_{\sup} \leqslant 1$ & $|\varphi^*(g_j)^{-1}|_{\sup} \leqslant 1$ factors uniquely through $\iota^* : A \to A\langle f, g^{-1}\rangle$.

$\varphi^* : A \to B$ extends to $A\langle\zeta, \xi\rangle \to B$, $\zeta_i \mapsto \varphi^*(f_i)$, $\xi_j \mapsto \varphi^*(g_j)^{-1}$, which descends to $A\langle f, g^{-1}\rangle \to B$, thus φ^* factors through ι^*.
Uniqueness: $A[g^{-1}]$ has dense image in $A\langle f, g^{-1}\rangle$.

Wts: $X\left(\frac{f}{f_0}\right) = \{x \in X : |f_i(x)| \leqslant |f_0(x)|\}$ is an aff'd subdomain.

$$A\langle\zeta\rangle := A\langle\zeta_1, \ldots, \zeta_r\rangle := \left\{\sum_\nu a_\nu \zeta^\nu \in A[[\zeta]] : a_\nu \in A, \lim a_\nu = 0\right\}$$

$$A\left\langle\frac{f}{f_0}\right\rangle := A\langle\zeta\rangle \Big/ (f_i - f_0\zeta_i : 1 \leqslant i \leqslant r)$$

Take $\iota^* : A \to A\left\langle\frac{f}{f_0}\right\rangle$ canonical; $\iota : X' := \operatorname{Sp} A\left\langle\frac{f}{f_0}\right\rangle \to X = \operatorname{Sp} A$.

Let $\varphi : Y = \operatorname{Sp} B \to X$, $\varphi^* : A \to B$.

$$\varphi(Y) \subseteq X\left(\frac{f}{f_0}\right) \Leftrightarrow \forall y, \forall i : |\varphi^*(f_i(y))| \leqslant |\varphi^*(f_0)(y)|$$

$$\Leftrightarrow \forall i : \left|\varphi^*(f_i)\varphi^*(f_0)^{-1}\right|_{\sup} \leqslant 1$$

Wts: $X\left(\frac{f}{f_0}\right) = \{x \in X : |f_i(x)| \leqslant |f_0(x)|\}$ is an aff'd subdomain.

$$A\langle\zeta\rangle := A\langle\zeta_1, \ldots, \zeta_r\rangle := \left\{\sum_\nu a_\nu \zeta^\nu \in A[[\zeta]] : a_\nu \in A, \lim a_\nu = 0\right\}$$

$$A\left\langle\frac{f}{f_0}\right\rangle := A\langle\zeta\rangle \Big/ (f_i - f_0\zeta_i : 1 \leqslant i \leqslant r)$$

Take $\iota^* : A \to A\left\langle\frac{f}{f_0}\right\rangle$ canonical; $\iota : X' := \operatorname{Sp} A\left\langle\frac{f}{f_0}\right\rangle \to X = \operatorname{Sp} A$.

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$$A\left\langle\frac{f}{f_0}\right\rangle := A\langle\zeta\rangle \Big/ (f_i - f_0\zeta_i : 1 \leqslant i \leqslant r)$$

Take $\iota^* : A \to A\left\langle\frac{f}{f_0}\right\rangle$ canonical; $\iota : X' := \operatorname{Sp} A\left\langle\frac{f}{f_0}\right\rangle \to X = \operatorname{Sp} A$.

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Take $\iota^* : A \to A\left\langle\frac{f}{f_0}\right\rangle$ canonical; $\iota : X' := \operatorname{Sp} A\left\langle\frac{f}{f_0}\right\rangle \to X = \operatorname{Sp} A$.

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$$\varphi(Y) \subseteq X\left(\frac{f}{f_0}\right) \Leftrightarrow \forall y, \forall i : |\varphi^*(f_i(y))| \leqslant |\varphi^*(f_0)(y)|$$

$$\Leftrightarrow \forall i : \left|\varphi^*(f_i)\varphi^*(f_0)^{-1}\right|_{\text{sup}} \leqslant 1$$

In particular if $Y = X'$, $\varphi = \iota$, we get $\iota(X') \subseteq X\left(\frac{f}{f_0}\right)$:

$$\iota^*(f_i) = \iota^*(f_0)\overline{\zeta_i} \Rightarrow \left|\iota^*(f_i)\iota^*(f_0)^{-1}\right|_{\text{sup}} = \left|\overline{\zeta_i}\right|_{\text{sup}} \overset{\text{contractivity}}{\leqslant} |\zeta_i|_{\text{sup}} = 1$$

$A\langle\zeta\rangle \twoheadrightarrow A\left\langle\frac{f}{f_0}\right\rangle$, $\zeta_i \mapsto \overline{\zeta_i}$ $\qquad \alpha : T_n\langle\zeta,\rangle \twoheadrightarrow A\langle\zeta\rangle \twoheadrightarrow A\left\langle\frac{f}{f_0}\right\rangle$

Wts: $X\left(\frac{f}{f_0}\right) = \{x \in X : |f_i(x)| \leqslant |f_0(x)|\}$ is an aff'd subdomain.

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$$A\left\langle\frac{f}{f_0}\right\rangle := A\langle\zeta\rangle \Big/ (f_i - f_0\zeta_i : 1 \leqslant i \leqslant r)$$

Take $\iota^* : A \to A\left\langle\frac{f}{f_0}\right\rangle$ canonical; $\iota : X' := \operatorname{Sp} A\left\langle\frac{f}{f_0}\right\rangle \to X = \operatorname{Sp} A$.

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$$\Leftrightarrow \forall i : \left|\varphi^*(f_i)\varphi^*(f_0)^{-1}\right|_{\sup} \leqslant 1$$

Universal property: wts $\forall \varphi^* : A \to B$ with $|\varphi^*(f_i)\varphi^*(f_0)^{-1}|_{\sup} \leqslant 1$ factors uniquely through $\iota^* : A \to A\left\langle\frac{f}{f_0}\right\rangle$.

$\varphi^* : A \to B$ extends to $A\langle\zeta\rangle \to B$, $\zeta_i \mapsto \varphi^*(f_i)\varphi^*(f_0)^{-1}$, which descends to $A\left\langle\frac{f}{f_0}\right\rangle \to B$, thus φ^* factors through ι^*.

Uniqueness: $A[f_0^{-1}]$ has dense image in $A\left\langle\frac{f}{f_0}\right\rangle$. □

Thank you for your attention!