Abstract Root Data

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Outline

The goal of this talk is to cover the following, as prescribed by the seminar programme:

- Define the notions of (reduced) root systems and root data, and discuss their relationship [Mil17, C.d, C.c].
- Classify root systems of rank 2, and explain how to associate a Cartan matrix and hence a Dynkin diagram to a root system [Mil17, C.g].
- Show that up to isomorphism, a root system is uniquely determined by its Dynkin diagram. State the classification result that the Dynkin diagrams of irreducible root systems are precisely the Dynkin diagrams A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4 and G_2 [Mil17, C.g].
- Give examples of reductive groups and their root data.

The main body of reference for these notes is [Mil17], in particular some parts of Appendix C as well as Chapter 21. For definitions, theorems etc. with a direct counterpart in that book, a reference of the shape C.⟨*number*⟩ is included. Typos in the book are highlighted in footnotes. Note that there is a new version of the book out there, published in 2022, but I did not manage to get my hands on it. Disclaimer: nothing here is my original work, except for possible typos, mistakes, misunderstandings and errors. I thank the organisers for their help and support.

Standing assumptions. All free modules will be of finite rank; in particular, vector spaces are always finite dimensional. F is a field of characteristic zero, V is a vector space over F.

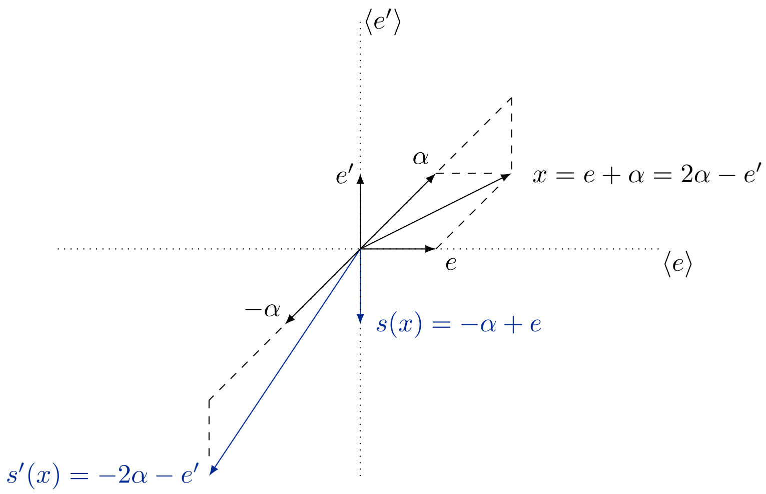
In practice, one can pretend that $F = \mathbb{Q}$ or $F = \mathbb{R}$ throughout the talk.

1 Root systems

Let $V^\vee = \mathrm{Hom}_F(V, F)$ denote the dual vector space, and let $\langle -, - \rangle : V \times V^\vee \to F$ be the natural pairing $(x, f) \mapsto f(x)$. For a vector $\alpha \in V$, we will let $\langle \alpha \rangle$ denote the line (subspace) generated by it. The two different uses of angle brackets should not lead to confusion.

Definition 1.1. A *reflection* is an $s \in \mathrm{Aut}(V)$ fixing the elements of some hyperplane and acting as (-1) on some complementary line. If $0 \neq \alpha \in V$ is an element of this line then s is a *reflection with vector α*. ○

Remark 1.2. A reflection is determined by a nonzero vector α *and* a hyperplane not containing it. However, the hyperplane is often suppressed from the notation. Consider the following example for two different reflections with the same vector but different hyperplanes:



Here $F = \mathbb{R}$, $\dim_{\mathbb{R}} V = 2$, and e resp. e' generate the hyperplanes $\langle e \rangle$ resp. $\langle e' \rangle$. These define the reflections s resp. s', which are clearly not the same.

Over $\mathbb{R}$, there is a unique *orthogonal* reflection with a given vector, but we won't need this. ∘

Lemma 1.3 (C.1)**.** *Let $0 \neq \alpha \in V$, $\alpha \notin H \subset V$ a hyperplane and $\alpha^\vee \in V^\vee$ such that $\alpha^\vee(H) = 0$ and $\langle \alpha, \alpha^\vee \rangle = 2$.* Define*

$$\begin{aligned} s_\alpha : V &\to V \\ x &\mapsto x - \langle x, \alpha^\vee \rangle \alpha \end{aligned}$$

Then s_α is a reflection with vector α. Conversely, for every reflection s with vector α there is a unique $\alpha^\vee$ such that s is of the form above. (In particular, suppressing $\alpha^\vee$ from the notation s_α makes sense.)

Proof. By assumption we have

$$\begin{aligned} s_\alpha(\alpha) &= \alpha - 2\alpha = -\alpha \\ \forall x \in H : s_\alpha(x) &= x - 0\alpha = x \end{aligned}$$

Conversely, the unique linear map with the prescribed properties is the composite

$$\begin{aligned} V \twoheadrightarrow V/H &\to F \\ \alpha + H &\mapsto 2 \end{aligned}$$

□

Lemma 1.4 (C.3)**.** *Let R be a finite spanning set for V, and $0 \neq \alpha \in V$. There is at most one reflection s with vector α such that $s(R) \subseteq R$.*

Proof. Assume s and s' are both such reflections (but possibly with different hyperplanes). Let $t := ss'$.

*The assumption $\alpha^\vee(H) = 0$ is missing in [Mil17].

On the one hand, $t(R) \subseteq R$. Since R is finite, it follows that there is some power $t^m = 1$. Hence the minimal polynomial of t divides $X^m - 1$.

On the other hand, t acts trivially on $V/\langle\alpha\rangle$, hence $(t-1)V \subseteq \langle\alpha\rangle$. Moreover, t also acts trivially on $\langle\alpha\rangle$, sending $\alpha \mapsto -\alpha \mapsto \alpha$. Hence $(t-1)^2 V \subseteq (t-1)\langle\alpha\rangle = 0$. In particular, the minimal polynomial of t divides $(X-1)^2$.

Since $X^m - 1$ and $(X-1)^2$ have greatest common divisor $X - 1$, we conclude that $t = 1$. □

Definition 1.5 (C.12–14). A subset $R \subseteq V$ is a *root system* if

(RS1) R is a finite spanning set for V not containing 0;

(RS2) $\forall \alpha \in R$, $\exists! s_\alpha$ reflection with vector α such that $s_\alpha(R) \subseteq R$

(RS3) $\forall \alpha, \beta \in R : s_\alpha(\beta) - \beta \in \mathbb{Z}\alpha$.

Note that we get uniqueness in RS2 for free from Lemma 1.4. An equivalent formulation for RS2 and RS3 is

(RS2+RS3) $\exists^\vee : R \to V^\vee$ such that $\forall \alpha \in R$, $\langle \alpha, \alpha^\vee \rangle = 2$, $\langle R, \alpha^\vee \rangle \subseteq \mathbb{Z}$, and $s_\alpha(R) \subseteq R$ where $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$.[†]

The functional $\alpha^\vee$ is the *coroot* of the *root* $\alpha \in R$. ∘

Definition 1.6 (C.15). The *Weyl group* $W(R)$ of a root system is the group generated by the reflections s_α. This action is faithful[‡] since R is a spanning set; since R is finite, so is $W(R)$.

For $\alpha \in R$, let $H_\alpha := \{x \in V : \langle x, \alpha^\vee \rangle = 0\}$ by the hyperplane complementary to $\langle\alpha\rangle$.[§] For $F \subseteq \mathbb{R}$, the connected components of $V - \bigcup_\alpha H_\alpha$ are called the *Weyl chambers*.

Note that F contains $\mathbb{Q}$, which is dense in $\mathbb{R}$, so these components will actually be the same regardless of F. I don't know what happens over $\mathbb{C}$. Over p-adic fields, this makes no sense, since they are totally disconnected, but we don't seem to care about these anyway. ∘

Let $c \in F^\times$, and assume that both α and $c\alpha$ are roots. Since $\langle \alpha, \alpha^\vee \rangle = 2$, we have that $\langle c\alpha, \alpha^\vee \rangle = 2c$ by linearity. Then RS3 implies that $2c \in \mathbb{Z}$. On the other hand, $\langle c\alpha, (c\alpha)^\vee \rangle = 2$, and thus $\langle \alpha, (c\alpha)^\vee \rangle = 2$ also has to be an integer by RS3. We conclude that

$$F\alpha \cap R = \begin{cases} \{\pm\alpha\} \\ \{\pm\alpha, \pm 2\alpha\} \\ \{\pm\alpha, \pm\frac{1}{2}\alpha\} \end{cases}$$

Definition 1.7 (C.18). A root system is *reduced* if $F\alpha \cap R = \{\pm\alpha\}$. ∘

Standing assumption. All root systems from now on will implicitly be understood to be reduced.

For $F \subseteq \mathbb{R}$, there is an inner product $(-,-)$ on V on which is preserved by $W(R)$ (in other words, elements of $W(R)$ act as orthogonal maps). Such an inner product can be constructed, for instance, by taking any inner product $(-,-)'$ on V and forcing it to become $W(R)$-invariant by summing over the orbits, using that $W(R)$ is finite: $(x, y) := \sum_{w \in W(R)} (wx, wy)'$.

[†]This formula has a typo in [Mil17]: the "$x-$" is missing.

[‡]In fact, the Weyl group acts transitively on Weyl chambers, see [Mil17, p. C.10].

[§]In [Mil17], H_α is confused with the corresponding subspace H'_α of the dual, resulting in a nonsensical definition of Weyl chambers.

Given an invariant inner product, we can be more explicit in our formulæ. In particular,

$$\langle x, \alpha^\vee \rangle = 2\frac{(x,\alpha)}{(\alpha,\alpha)} \tag{1}$$

$$s_\alpha(x) = x - 2\frac{(x,\alpha)}{(\alpha,\alpha)}\alpha \tag{2}$$

Definition 1.8 (C.21). A *base* for a root system over $F \subseteq \mathbb{R}$ is a subset $S \subseteq R$ such that

- S is a basis for V and
- each root in R can be written as a linear combination of elements of S with integer coefficients of the same sign (that is, either non-negative or non-positive coefficients only).

For a given base, elements of the base expressible as a non-negative linear combination are called *positive elements* for that base. ◦

Proposition 1.9 (C.9, C.22). *Let $F \subseteq \mathbb{R}$. Let t be an element of some Weyl chamber. Then the indecomposable elements of $\{\alpha \in R : (\alpha, t) > 0\}$ form a base of the root system, and the positive roots with respect to this base are exactly $\{\alpha \in R : (\alpha, t) > 0\}$. Conversely, every base arises this way from a unique Weyl chamber.*

Proof. ↯ Suppose that the set S_t of indecomposable elements of $\{\alpha \in R : (\alpha, t) > 0\}$ is not a base. By definition, there will be some $\alpha \in R$ not a linear combination of elements of S_t with integer coefficients of the same sign. By perhaps replacing α by $-\alpha$, we may assume that $(\alpha, t) > 0$ and it cannot be written as a linear combination of elements of S_t with non-negative integer coefficients. Since R is finite, we may choose such an α so that $(\alpha, t) > 0$ is minimal.

By definition of S_t, α must be decomposable, so we have $\alpha = \beta + \gamma$ where $(\beta, t) > 0$ and $(\gamma, t) > 0$. Then $(\alpha, t) = (\beta, t) + (\gamma, t)$ by linearity of the inner product, hence both terms on the right hand side are strictly smaller than (α, t). It follows from minimality that β and γ are linear combinations of elements of S_t with non-negative integer coefficients – but then so is α. ↯

We won't prove the converse. To give an idea of what happens: let S be a base for R. The corresponding Weyl chamber will be $C := \{t \in V : \forall \alpha \in S, (\alpha, t) > 0\}$; in other words, this is the intersection of the positive half-spaces determined by H_α for $\alpha \in S$. On the one hand, this is an open convex cone, so in particular it is connected with boundary $\partial C = \bigcup_{\alpha \in S} H_\alpha$. It remains to be shown that C is indeed a connected component of $V - \bigcup_{\alpha \in R} H_\alpha$, that is, no H_α goes through C. A proof is presented in [Hum90, Theorem 1.12(a)], where it is shown that C (or rather its closure) is a fundamental domain for the action of $W(R)$ (which is equivalent to no H_α passing through C). This, in turn, is proven through an analysis of $W(R)$ stretching over multiple sections, and proving much more about $W(R)$ than we need here. I shied away from dissecting the proof to see what's needed for just this statement; from what I can tell, the steps are rather elementary. Milne calls the whole proof straightforward. □

Definition 1.10 (C.23). Let $\{(V_i, R_i) : i \in I\}$ be a finite family of root systems. We define their *direct sum* to be

$$\bigoplus_{i \in I}(V_i, R_i) := \left(\bigoplus_{i \in I} V_i, \bigsqcup_{i \in I} R_i\right)$$

This is also a root system. A root system that cannot be written as a direct sum of nonempty root systems is called *indecomposable* or *irreducible*. ◦

Remark. Note that the property of being indecomposable has nothing to do with the root system being reduced. I will shun the term irreducible to avoid confusion. (Not being an algebraic geometer, I can say that I don't like it when something is both reduced and irreducible.) There are many words out there: let's not overload their meanings. ◦

This is a good point to stop and illustrate the above cornucopia of definitions through some examples. In dimension 1, the only root system is A_1, which is just $\{\pm 1\}$ on a line; there is no Weyl chamber. Figure 1 presents all dimension 2 reduced root systems for $F = \mathbb{R}$. The thick vectors form the base corresponding to the shaded Weyl chamber. The root system $A_1 \times A_1$ is decomposable into the direct sum of two A_1's, hence the notation.

2 Root data

Let X be a free $\mathbb{Z}$-module (of finite rank). Let $X^\vee := \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ denote its dual, and let $\langle -, - \rangle : X \times X^\vee \to \mathbb{Z}$ denote the natural pairing.

Definition 2.1 (C.28). A *root datum* is a triple $\mathcal{R} = (X, R, ^\vee)$ where $R \subset X$ is a finite subset, and $^\vee : R \to X^\vee$ such that the following hold:

(RD1) $\forall \alpha \in R : \langle \alpha, \alpha^\vee \rangle = 2$;

(RD2) $\forall \alpha \in R : s_\alpha(R) \subseteq R$ where $s_\alpha : x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$;

(RD3) $W(\mathcal{R}) = \langle s_\alpha : \alpha \in R \rangle \leq \mathrm{Aut}(X)$ is finite.

We call $W(\mathcal{R})$ the *Weyl group*. Elements of R resp. $R^\vee$ are the *roots* resp. *coroots*. ◦

There is an apparent similarity between the definitions of root systems and root data. We will now explore their relationship. First of all, notice that s_α is a reflection with vector α in the vector space $X \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 2.2. A root datum is *reduced* if $\mathbb{Z}\alpha \cap R = \pm\alpha$. (We always have $-\alpha = s_\alpha(\alpha) \in R$ by RD2 and RD1.) ◦

Definition 2.3 (C.36). The *rank* of a root datum is $\dim_{\mathbb{Q}}(X \otimes_{\mathbb{Z}} \mathbb{Q})$. The *semisimple rank* is $\dim_{\mathbb{Q}}(\mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{Q})$. ◦

Standing assumption. From now on, all root data are reduced.

Proposition 2.4 (C.29). *Let $(X, R, ^\vee)$ be a triple satisfying (RD1) and (RD2), and let $V := (\mathbb{Z}R) \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq X \otimes_{\mathbb{Z}} \mathbb{Q}$. Then (V, R) is a not necessarily reduced root system. Moreover, the image of $\alpha^\vee$ under the map $X^\vee \to V^\vee$ induced by $V \hookrightarrow X$ is the root-system-theoretic coroot of α.*

Proof. (RS1) is fulfilled by definition of V. We now check that (RS2+RS3) holds. For this, we need a map $^{\mathrm{v}} : R \to V^\vee$; to avoid confusion, in this proof we shall use $^\vee$ to denote the root datum map and $^{\mathrm{v}}$ the root system map to avoid overloading the operators too much. We define the latter to be the composite

$$^{\mathrm{v}} : R \to X^\vee \to (X \otimes_{\mathbb{Z}} \mathbb{Q})^\vee \twoheadrightarrow V^\vee$$

Here the first arrow is the map from the root datum, the second one is induced by $X \to X \otimes_{\mathbb{Z}} \mathbb{Q}, x \mapsto x \otimes 1$, while the third one is induced by the inclusion $V = \mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow X \otimes_{\mathbb{Z}} \mathbb{Q}$.

The conditions that $\langle \alpha, \alpha^{\mathrm{v}} \rangle = 2$ resp. $s_\alpha(R) \subseteq R$ are satisfied due to (RD1) resp. (RD2). Finally, $\langle R, \alpha^{\mathrm{v}} \rangle = \langle R, \alpha^\vee \rangle$ is contained in $\mathbb{Z}$ since $\alpha^\vee \in X^\vee = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$. □

In particular, a root datum always gives rise to a root system.

Definition 2.5 (C.34)**.** A root datum $(X, R, {}^\vee)$ is called *semisimple* if R spans $X \otimes_{\mathbb{Z}} \mathbb{Q}$. ∘

So a semisimple root datum $(X, R, {}^\vee)$ gives rise to a root system $(X \otimes_{\mathbb{Z}} \mathbb{Q}, R)$ via Proposition 2.4. To make the relationship between root data and root systems more precise, we need to make one final batch of definitions.

Definition 2.6 (C.26)**.** Let (V, R) be a root system. A subgroup $X \subseteq V$ is called a *lattice* if as a $\mathbb{Z}$-module it is generated by a basis of V.[¶] We define the *root lattice* $Q(R) := \mathbb{Z}R$ and the *weight lattice* $P(R) := \{v \in V : \forall \alpha \in R, \langle v, \alpha^\vee \rangle \in \mathbb{Z}\}$ in V. Note that $Q(R) \subseteq P(R)$ by (RS3). ∘

Proposition 2.7 (C.35)**.** *If (X, R) is a semisimple root datum, then $(X \otimes_{\mathbb{Z}} \mathbb{Q}, R)$ is a root system with the same map $\alpha \mapsto \alpha^\vee$, and $Q(R) \subseteq X \subseteq P(R)$.*

Conversely, for any root system (V, R) and any lattice X in V such that $Q(R) \subseteq X \subseteq P(R)$, the triple $(X, R, {}^\vee)$ is a semisimple root datum.

Proof. Proposition 2.4 tells us that $(X \otimes_{\mathbb{Z}} \mathbb{Q}, R)$ is a root system. We have $Q(R) = \mathbb{Z}R \subseteq X$ since $R \subseteq X$. We also have $X \subseteq P(R)$ because $R^\vee \subseteq X^\vee = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$.

Conversely, let (V, R) and X be as above. For all $\alpha \in R$, we have $\langle R, \alpha^\vee \rangle \subseteq \mathbb{Z}$ by (RS3), and thus $\alpha^\vee \in X^\vee$, so we have a map ${}^\vee : R \to X^\vee$ as required. Conditions (RD1) and (RD2) follow directly from (RS2+RS3). As already noted in Definition 1.6, $\langle s_\alpha : \alpha \in R \rangle \leq \mathrm{Aut}(V)$ acts faithfully on R, hence it is finite. Hence the same holds for $\langle s_\alpha : \alpha \in R \rangle \leq \mathrm{Aut}(X)$, verifying (RD3). Thus we indeed have a root datum; semisimplicity is due to X being a lattice in V. □

3 Classification of reduced root systems

Definition 3.1 ([Hum72, p. 43])**.** Two root systems (V, R) and (V', R') are called *isomorphic* if there exists a vector space isomorphism $\varphi : V \xrightarrow{\sim} V'$ such that $\varphi(R) = R'$ and for all $\alpha, \beta \in V$, $\langle \beta, \alpha^\vee \rangle = \langle \varphi(\beta), \varphi(\alpha)^\vee \rangle$. ∘

3.1 In rank 1

There is nothing to do: up to isomorphism (that is, scaling), the only reduced root system is A_1.

A_1

3.2 In rank 2

Assume $F = \mathbb{R}$. As we have seen in (1), given an invariant inner product, we have

$$\langle \beta, \alpha^\vee \rangle = 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} = 2\frac{|\beta|}{|\alpha|}\cos\varphi$$

where φ is the angle between the vectors α and β. By (RS2+RS3) we know that this number is an integer. Exploiting symmetry, we obtain that

$$\langle \beta, \alpha^\vee \rangle \langle \alpha, \beta^\vee \rangle = 4\cos^2\varphi$$

In order for this to be an integer, we must have $\cos\varphi \in \{0, \pm\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \pm 1\}$. We want α and β to be elements of a base of a reduced root system; in particular, one is not a multiple of the other.

$\langle\beta,\alpha^\vee\rangle\langle\alpha,\beta^\vee\rangle$	$\langle\alpha,\beta^\vee\rangle$	$\langle\beta,\alpha^\vee\rangle$	φ	$\lvert\beta\rvert/\lvert\alpha\rvert$	root system
0	0	0	$\pi/2$		$A_1 \times A_1$
1	1	1	$\pi/3$	1	—
	-1	-1	$2\pi/3$		A_2
2	1	2	$\pi/4$	$\sqrt{2}$	—
	-1	-2	$3\pi/4$		B_2
3	1	3	$\pi/6$	$\sqrt{3}$	—
	-1	-3	$5\pi/6$		G_2

Table 1: List of cases.

Furthermore, without loss of generality we may assume that $|\beta| \geq |\alpha|$. Now using that cos is bijective on $[0, \pi]$, we conclude that we must be in one of the cases listed in Table 1.

If $\cos\varphi > 0$ (φ is acute) then we see that at least one of $\langle\beta,\alpha^\vee\rangle$ or $\langle\alpha,\beta^\vee\rangle$ is 1. Say it's the former; the argument will be completely symmetric. Now using (2), we find that $s_\alpha(\beta) = \beta - \langle\beta,\alpha^\vee\rangle\alpha = \beta - \alpha$, therefore $\pm(\beta - \alpha) \in R$. Since we assumed $\{\alpha, \beta\}$ to be a base, this means that one of them, say, $\beta - \alpha$ is a linear combination. But then $\beta = (\beta - \alpha) + \alpha$ is not an indecomposable element, which contradicts Proposition 1.9 (we need to use both directions of the statement).

Consequently, $\cos\varphi \leq 0$. The remaining four rows of our table are realised by the root systems in Figure 1. It is clear that they are not isomorphic to each other since $\langle\beta,\alpha^\vee\rangle$ is different for each of them.

3.3 Cartan matrices and Dynkin diagrams

As we have just witnessed, the integrality condition on $\langle\beta,\alpha^\vee\rangle$ is a powerful tool in classifying root systems.

Definition 3.2. The *Cartan matrix* of a root system is the matrix $(\langle\beta,\alpha^\vee\rangle)_{\alpha,\beta\in S}$ where S is a base. ◦

Remark 3.3. By (RS2+RS3) we know that the diagonal entries of a Cartan matrix are 2's. The off-diagonal entries are non-positive: indeed, for any two distinct base elements α, β, they span a 2-dimensional subspace in which they form a base to a root system of rank 2; now use the same argument as in the classification of rank 2 systems. ◦

Note that Cartan matrices are isomorphism invariant. In fact, the converse is also true:

Proposition 3.4 (C.52). *The Cartan matrix is independent of the base (up to reordering),*[‖] *and determined the root system up to isomorphism.*

Proof. We omit the proof. The underlying idea is that bases correspond to Weyl chambers, upon which the Weyl group acts simply transitively; at the same time, the Weyl group action preserves $(\langle\beta,\alpha^\vee\rangle)_{\alpha,\beta\in S}$. Using this, given two root systems with the same Cartan matrix, one can construct an isomorphism. □

Example 3.5. The Cartan matrices of rank 2 reduced root systems are shown in Table 2. ◦

[¶]Some authors use the term *lattice* to mean any finitely generated torsionfree module over an integral domain; then the notion defined above is that of a *full* $\mathbb{Z}$*-lattice*. See [Rei03, p. 44].

[‖]This obvious remark is missing in [Mil17].

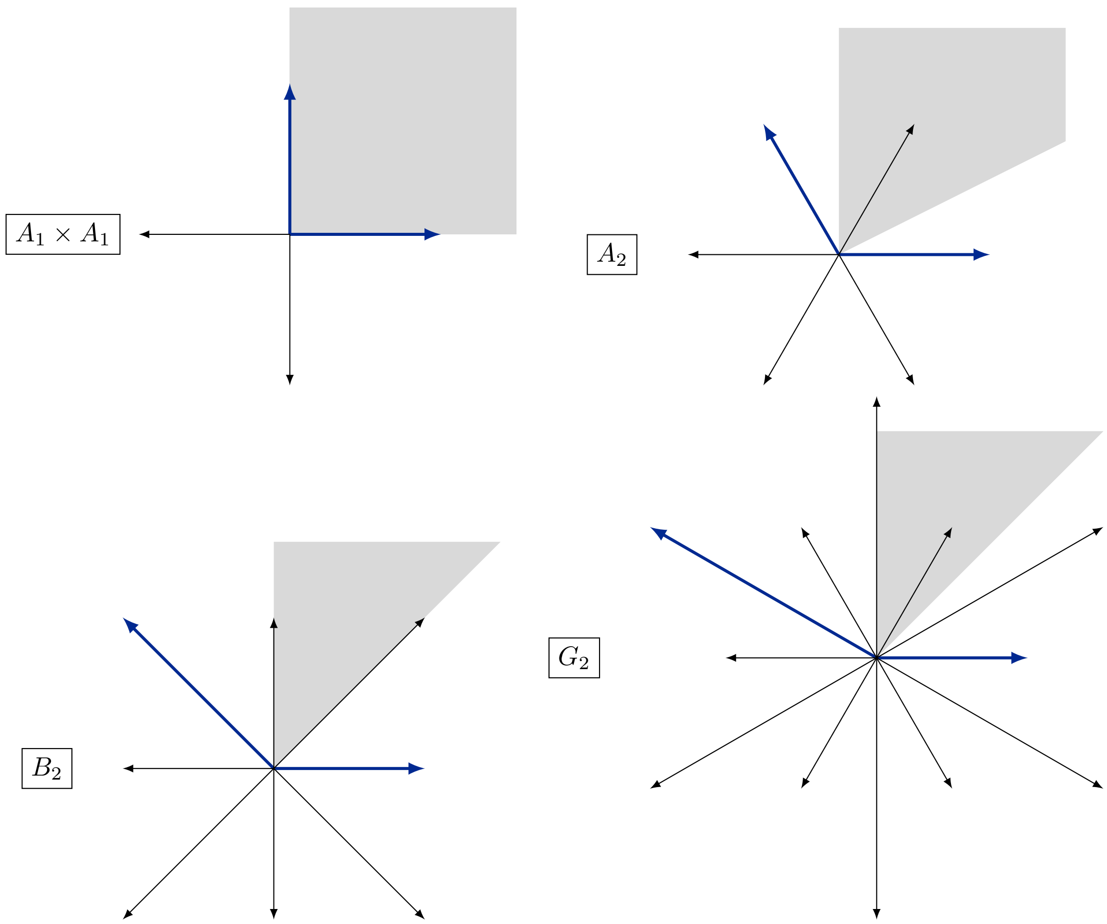


Figure 1: Rank 2 reduced root systems.

$$A_1 \times A_1\colon \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad A_2\colon \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$
$$B_2\colon \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \quad G_2\colon \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Table 2: Cartan matrices of rank 2 reduced root systems

Another way to harness the power of $(\langle\beta,\alpha^\vee\rangle)_{\alpha,\beta\in S}$ for classification purposes is the introduction of Dynkin diagrams.

Definition 3.6. Let S be a base for the root system (V, R). The *Coxeter graph* has vertices the elements of S, and $\alpha, \beta \in S$ are joined by $\langle\beta,\alpha^\vee\rangle\langle\alpha,\beta^\vee\rangle$ (undirected) edges.

The *Dynkin diagram* is the Coxeter graph with an inequality sign "$<$" pointing towards α if $|\alpha| < |\beta|$. ∘

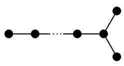
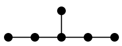
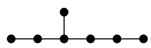
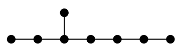
The Coxeter graph is in general not fine enough to distinguish between root systems up to isomorphism (it is in ranks 1 and 2). The Dynkin diagram fares better:

Proposition 3.7. *The Dynkin diagram determines the Cartan matrix (and hence the root system).*

Proof. Let α, β be two distinct base elements. In the subspace spanned by them, they form a base to a root system of rank 2. Using Table 1 *and* information of which root is longer, we recover the Cartan matrix.

This can also be shown directly, see [Hal15, Proposition 8.32]. □

Theorem 3.8 (C.56)**.** *Dynkin diagrams of indecomposable root systems are as follows:*

- A_n *for* $n \geq 1$*:*
- B_n *for* $n \geq 2$*:*
- C_n *for* $n \geq 3$*:*
- D_n *for* $n \geq 4$*:* 
- E_6*:* 
- E_7*:* 
- E_8*:* 
- F_4*:*
- G_2*:*

4 Examples of reductive groups and their root data

Before turning our attention to examples, we recall how one can associate a root datum with a split reductive group: we have seen all this in the previous talk. This practically gives us a recipe for turning a reductive group with a maximal torus into a root datum, which then gives rise a root system:

$$\begin{matrix}\text{reductive group}\\ \text{with maximal torus}\end{matrix} \rightsquigarrow \text{root datum} \overset{-\otimes_{\mathbb{Z}}\mathbb{Q}}{\rightsquigarrow} \text{root system}$$

As we have seen above, root systems possess a nice classification result, which is why attaching a root datum to a reductive group is a good idea.

Let (G, T) be a pair where G is a reductive group and T is a maximal torus.

Remark 4.1. Milne talks about split reductive groups instead: these are pairs (G, T) where G is a reductive group and T is a split maximal torus, that is, a maximal torus isomorphic to a product of copies of $\mathbb{G}_m$. Since we always work over algebraically closed fields, for us every torus is split. ∘

Let $\mathfrak{g}$ be the Lie algebra of G, and let $G \to \mathrm{GL}_\mathfrak{g}$ be the adjoint representation (i.e. the action is via inner automorphisms); for details, see [Mil17, §§10.b, 10.d]. This descends to an action of

T on the Lie algebra $\mathfrak{g}$. The (split maximal) torus T is diagonalisable, and hence (consult the Theorem on [Talk 6, p. 19]) there is a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_0 = \mathfrak{g}^T = \mathfrak{t}$ is the subspace fixed by T, which also agrees with the Lie algebra $\mathfrak{t}$ of T, and $\mathfrak{g}_\alpha$ is the subspace on which T acts via the character $\alpha : T \to \mathbb{G}_m$ (recall that $X^*(T)$ denotes the group of characters of T). It can be shown [Mil17, p. 425] that $\mathfrak{g}_0 = \mathfrak{t}$. We let $\Phi(G,T) \subseteq X^*(T)$ denote the subset of characters occurring in the decomposition above.

The Weyl group is defined as $W(G,T) := N_G(T)/T$, see [Talk 10, Definition 1.5, Proposition 4.5]. This is a finite constant algebraic group by [Mil17, Proposition 21.1]. The Weyl group acts on $\Phi(G,T)$ such that $s\mathfrak{g}_\alpha = \mathfrak{g}_{s\alpha}$ for all $s \in W(G,T)$.

Remark 4.2. The equation $s\mathfrak{g}_\alpha = \mathfrak{g}_{s\alpha}$ above is an abuse of notation. Indeed, $W(G,T)$ does not act on $\mathfrak{g}$. What is true, however, is that if $n \in N_G(T)$ is a lift of s then n does act on $\mathfrak{g}$ via the adjoint action, but this action depends on the choice of s. Any such n sends eigenspaces to eigenspaces, that is, it acts on the set of eigenspaces, and it is this action which is independent of the choice. Since eigenspaces correspond to $\Phi(G,T)$, we get the action of $W(G,T)$ on the roots. ∘

Recall from [Talk 6, pp. 6–7] that characters $X^*(T) = \{T \to \mathbb{G}_m\}$ form a free $\mathbb{Z}$-module. Indeed, the group is obviously abelian; and it is isomorphic to the group of group-like elements in $\mathcal{O}(G)$, which are linearly independent, hence the freeness.

With notation as above, we let $T_\alpha := \ker(\alpha) \subseteq T$ and $G_\alpha := C_g(T_\alpha) \supseteq C_G(T) = T$. Then the following hold ([Mil17, Theorem 21.11]:

- $\mathbb{Q}\alpha \cap \Phi(G,T) = \{\pm\alpha\}$
- (G_α, T) is split reductive with $W(G_\alpha, T) = \{1, s_\alpha\}$ for some nontrivial s_α
- $\exists! \alpha^\vee \in X_*(T)$ cocharacter such that $\langle \alpha, \alpha^\vee \rangle = 2$ and $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ for all $x \in X^*(T)$

The pairing in the last statement is the pairing of characters with cocharacters obtained by the composition $\mathbb{G}_m \xrightarrow{\alpha^\vee} T \xrightarrow{\alpha} \mathbb{G}_m$. Note that $\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$.

Combining all of the above, it follows that $(X^*(T), \Phi(G,T), ^\vee)$ is a reduced root datum, see [Mil17, Corollary 21.12].

To showcase what root data of reductive groups are capable of, we state the following result without proof. For $\alpha \in \Phi(G,T)$, let $H_\alpha := \{f \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q} : \langle \alpha, f \rangle = 0\}$: this is a hyperplane in $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. The *Weyl chambers* of (G,T) are defined to be the connected components of $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q} - \bigcup_{\alpha \in \Phi(G,T)} H_\alpha$.

Theorem 4.3 **([Mil17, pp. 21.31, 21.32]).** *Let $\lambda, \lambda' \in X_*(T)$. They lie in the same Weyl chamber iff $P(\lambda) = P(\lambda')$. There is an isomorphism*

$$\{\textit{Weyl chambers of } (G,T)\} \to \{T \subseteq B \subseteq G : B \textit{ Borel}\}$$
$$\lambda \mapsto P(\lambda)$$

The $P(\lambda)$ here was defined in [Talk 11]. The proof is not difficult, given what we already know about Borel subgroups, but omitted for time reasons. Instead, we discuss some examples.

Example 4.4 ([Mil17, pp. 21.3, 21.6, 21.16]). Let $G = \mathrm{GL}_2$ and $T := \mathbb{D}_2 := \left\{ \begin{pmatrix} k^\times & 0 \\ 0 & k^\times \end{pmatrix} \right\} \simeq \mathbb{G}_m \times \mathbb{G}_m$ be the diagonal torus. The character group of T is generated by the two characters $\chi_1 : \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \mapsto$

t_1 and $\chi_2 : \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right) \mapsto t_2$, thus $X^*(T) = \mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_2$ with $(m_1\chi_1 + m_2\chi_2)\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right) = t_1^{m_1} t_2^{m_2}$. Indeed, we have seen that $X^*(\mathbb{G}_m) \simeq \mathbb{Z}$, therefore $X^*(\mathbb{G}_m \times \mathbb{G}_m) \simeq \mathbb{Z} \times \mathbb{Z}$.

The Lie algebra of GL_2 is $\mathfrak{gl}_2 = M_2(k)$, with the Lie bracket being the additive commutator $[A, B] = AB - BA$. The adjoint action is via conjugation:

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t_1^{-1} & 0 \\ 0 & t_2^{-1} \end{pmatrix} = \begin{pmatrix} a & \frac{t_1}{t_2} b \\ \frac{t_2}{t_1} c & d \end{pmatrix}$$

Thus $\mathfrak{gl}_2$ decomposes into the following subspaces:

$$\mathfrak{gl}_2 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \oplus \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} = \mathfrak{t} \oplus \mathfrak{g}_{\chi_1 - \chi_2} \oplus \mathfrak{g}_{\chi_2 - \chi_1}$$

Thus the roots of the split reductive group (GL_2, T) are $\Phi(\mathrm{GL}_2, T) = \{\pm(\chi_1 - \chi_2)\}$.

The split reductive group $(\mathrm{GL}_n, \mathbb{D}_n)$ with $\mathbb{D}_n$ again being the diagonal torus can be treated in a completely analogous fashion. There are n linearly independent characters χ_i picking out the ith element from the diagonal, and the roots are their differences: $\Phi(\mathrm{GL}_n, T) = \{\chi_i - \chi_j : i \neq j\}$; the adjoint action is $\chi_i - \chi_j$ on the (i, j)th entry of a matrix.

Now we will determine the coroots. Let $\alpha := \chi_1 - \chi_2$ be a root. Then by definition,

$$T_\alpha = \ker(\alpha) = \{\mathrm{diag}(x, x, x_3, \ldots, x_n) : x x x_3 \ldots x_n \neq 0\}$$

Its centraliser is the group

$$G_\alpha = \left\{ \begin{pmatrix} * & * & 0 & \ldots & 0 \\ * & * & 0 & \ldots & 0 \\ 0 & 0 & * & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & * \end{pmatrix} \in \mathrm{GL}_n \right\}$$

We know that the Weyl group $W(G_\alpha, T) = N_{G_\alpha}(T)/T$ is of order 2; a representative of the nontrivial element s_α is

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix} : \mathrm{diag}(x_1, x_2, x_3, \ldots, x_n) \mapsto \mathrm{diag}(x_2, x_1, x_3, \ldots, x_n)$$

Therefore s_α acts on $\chi = m_1\chi_1 + \ldots + m_n\chi_n \in X^*(T)$ as

$$s_\alpha(\chi) = m_2\chi_1 + m_1\chi_2 + m_3\chi_3 + \ldots + m_n\chi_n = \chi - \langle \chi, \lambda_1 - \lambda_2 \rangle (\chi_1 - \chi_2)$$

where λ_1 resp. λ_2 are the cocharacters

$$\lambda_1 : t \mapsto \mathrm{diag}(t, 1, 1, \ldots, 1); \quad \lambda_2 : t \mapsto \mathrm{diag}(1, t, 1, \ldots, 1)$$

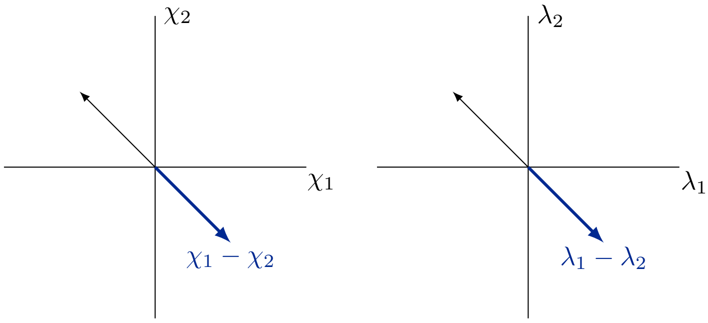
It follows that the corresponding coroot is

$$\alpha^\vee = \lambda_1 - \lambda_2 : t \mapsto \mathrm{diag}(t, t^{-1}, 1, \ldots, 1)$$

Similarly for $\chi_i - \chi_j$.

After tensoring with $\mathbb{Q}$, we obtain a root system (Proposition 2.7). For $n = 2$ the root datum has semisimple rank 1, and the root system is A_1, see Figure 2. For $n = 3$, see Figure 3.

◦



$\Phi(\mathrm{GL}_2, T)$ $\Phi(\mathrm{GL}_2, T)^\vee$

Figure 2: Root system A_1 associated with GL_2.

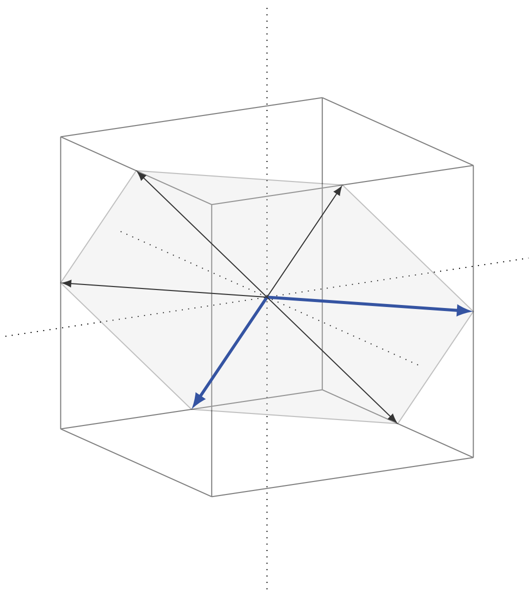


Figure 3: Root system A_2 associated with GL_3. A basis is $\{\chi_1 - \chi_2, \chi_2 - \chi_3\}$.

Example 4.5 ([Mil17, pp. 456–457]). Let $n \geq 1$, $G = \mathrm{SL}_{n+1}$ and $T = \{\mathrm{diag}(t_1, \dots, t_{n+1}) : t_1 \dots t_{n+1} = 1\}$. It has characters $\chi_i : \mathrm{diag}(t_1, \dots, t_{n+1}) \mapsto t_i$ for $i = 1, \dots, n+1$ with the single relation $\chi := \chi_1 + \dots + \chi_{n+1} = 0$ between them, hence the character group is

$$X^*(T) = \bigoplus_{i=1}^{n+1} \mathbb{Z}\chi_i \Big/ \mathbb{Z}\chi$$

Dually, the group of cocharacters is

$$X_*(T) = \left\{ \sum_{i=1}^{n+1} m_i \lambda_i : \sum_{i=1}^{n+1} m_i = 0 \right\}$$

The Lie algebra of SL_{n+1} is $\mathfrak{sl}_{n+1} = \{M \in M_{n+1}(k) : \mathrm{Tr}\, M = 0\}$. Analogously to the $\mathfrak{gl}_n$ case above, we have that the adjoint action is via $\chi_i - \chi_j$ in the (i,j)th entry. Denoting the equivalence class of χ_i in $X^*(T)$ by $\bar{\chi}_i$, this yields the following decomposition:

$$\mathfrak{sl}_{n+1} = \mathfrak{g}_0 \oplus \bigoplus_{i \neq j} \mathfrak{g}_{\bar{\chi}_i - \bar{\chi}_j}$$

Thus the roots are $\Phi(\mathrm{SL}_{n+1}, T) = \{\bar{\chi}_i - \bar{\chi}_j : i \neq j\}$.

We again restrict ourselves to computing the coroot of $\alpha = \bar{\chi}_1 - \bar{\chi}_2$; the general case is entirely similar. We have

$$T_\alpha = \ker(\alpha) = \{\mathrm{diag}(x, x, x_3, \dots, x_n) : x x x_3 \dots x_n \neq 0\}$$

$$G_\alpha = \left\{ \begin{pmatrix} * & * & 0 & \dots & 0 \\ * & * & 0 & \dots & 0 \\ 0 & 0 & * & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & * \end{pmatrix} \in \mathrm{SL}_{n+1} \right\}$$

The nontrivial element s_α of $W(G_\alpha, T)$ is represented by

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

We need the minus sign so that the determinant remains $+1$. Let $\psi = \sum_{i=1}^{n+1} a_i \bar{\chi}_i \in X^*(T)$. Then s_α acts as

$$s_\alpha(\psi) = a_2 \bar{\chi}_1 + a_1 \bar{\chi}_2 + \sum_{i=3}^{n+1} a_i \bar{\chi}_i = \psi - \langle \psi, \lambda_1 - \lambda_2 \rangle (\bar{\chi}_1 - \bar{\chi}_2)$$

where λ_1 and λ_2 are as above. Therefore $\alpha^\vee = \lambda_1 - \lambda_2$.

For SL_2, the resulting root system is depicted in Figure 4. The roots and coroots are:

$$\Phi(\mathrm{SL}_2, T) = \{\pm \bar{\chi}_1 \mp \bar{\chi}_2\} = \{\pm 2\bar{\chi}_1\}$$
$$\Phi(\mathrm{SL}_2, T)^\vee = \{\pm \lambda_1 \mp \lambda_2\} = \{\pm \lambda_1\}$$

After tensoring with $\mathbb{Q}$, we again obtain a root system (Proposition 2.7). A base is given by $\{\bar{\chi}_1 - \bar{\chi}_2, \dots, \bar{\chi}_n - \bar{\chi}_{n+1}\}$. ◦

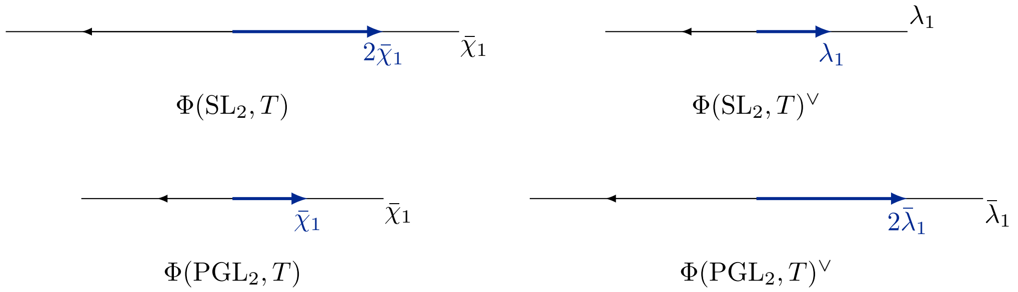


Figure 4: Root systems associated with SL_2 and PGL_2.

Example 4.6. For PGL_2, the computation is similar to the one above because the adjoint action is the same as for GL_2. The character group is generated by $\bar{\chi}_1 : [\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)] \mapsto t$, the cocharacer group by $\bar{\lambda}_1 : t \mapsto [\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)]$. The coroot corresponding to $\bar{\chi}_1$ is $2\bar{\lambda}_1$. See Figure 4. ∘

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