Deformations of Galois representations II: Lifting Galois representations

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This talks is mostly based on Barry Mazur's paper [Maz97]. These notes were produced in preparation for the talk; apart from the typos, mistakes, and errors, nothing here is my intellectual product. Questions, comments, and suggestions for improvement are welcome. I would like to extend my gratitude to Andreas, Antonio, Luca, Ludvig, Manuel, and Nicolas (in alphabetical order) for helpful discussions.

In Mazur's paper, 'ring' appears to mean 'commutative ring with unity'.

0 Introduction

In Section 1, we will explain why we study Galois representations. Then in Section 2, we will use the power of deformation theory to obtain number-theoretic results. The rest of the script concerns so-called deformation conditions, as explained in Section 3.

In this script, we will consider:

- K a number field (finite extension of $\mathbb{Q}$)
- N a positive integer
- Π a profinite group that is p-finite (i.e. has an open subgroup Π_o of finite index such that $\mathrm{Hom}_{\mathrm{cts}}(\Pi_o, \mathbb{Z}/p\mathbb{Z})$ is finite) – in applications, this will be some absolute Galois group
- k a finite field or characteristic p
- Λ a coefficient-ring (i.e. complete local noetherian ring with finite residue field) with residue field k
- $\bar{\rho} : \Pi \to \mathrm{GL}_N(k)$ continuous representation
- $\bar{V} := k^N$ with k-linear Π-action via $\bar{\rho}$
- A a coefficient-Λ-algebra (i.e. a coefficient-ring and a Λ-algebra)
- $\rho : \Pi \to \mathrm{GL}_N(A)$ a lift of $\bar{\rho}$
- $V(\rho) := A^N$ with Π-action via $\bar{\rho}$

1 Local and global absolute Galois groups

Reference: [Maz97, §1].

1.1 The local–global package

Recall that we are trying to study something like an absolute Galois group. More precisely, let K be a number field (i.e. a finite extension of $\mathbb{Q}$), K^{al} an algebraic closure, S a finite set of non-archimedean (i.e. finite) primes of K. We let K_S^{al} be the maximal algebraic extension of K inside K^{al} unramified outside S. The extension K_S^{al}/K is automatically Galois, and we want to study the Galois group $G_{K,S} := \mathrm{Gal}(K_S^{\mathrm{al}}/K)$.

The group $G_{K,S}$ is profinite: since K_S^{al} is the union of all finite extensions of K unramified outside S (equivalently: with discriminant coprime to S), the group $G_{K,S}$ is the inverse limit of all Galois subextensions. In particular, $G_{K,S}$ carries the natural Krull topology. We have seen in Luca's talk that it is also p-finite, which is why in the above, we always assume our Π to be profinite and p-finite.

The global group $G_{K,S}$ is difficult to understand, which is why one usually tries studying local versions as well. More explicitly, for any place v of K, let K_v denote the corresponding completion, which is an (archimedean or nonarchimedean) local field. For each v, fix a (non-canonical) embedding $\iota_v : K^{\mathrm{al}} \to K_v^{\mathrm{al}}$. This gives rise to continuous homomorphisms $i_v : G_{K_v} := \mathrm{Gal}(K_v^{\mathrm{al}}/K_v) \to G_{K,S}$ given by post-composition with ι_v. Changing ι_v induces replacing i_v by a conjugate of it.

Recollection. Before the next step, let us recall the definition of an inertia subgroup in the global and local context. In the global case, let L/K be a number field Galois extension and v a nonarchimedean place, the inertia subgroup of the Galois group $\mathrm{Gal}(L/K)$ is

$$I_{L/K,v} := \{\sigma \in \mathrm{Gal}(L/K) : \sigma(v) = v \text{ and } \forall \alpha \in L, \sigma(\alpha) \equiv \alpha \bmod v\}$$

In the local case, fix a prime $w \mid v$ of L above v: then we have a Galois extension of the completions L_w/K_v, and the inertia group is

$$I_{L_w/K_v} := \{\sigma \in \mathrm{Gal}(L_w/K_v) : \forall \alpha \in L_w, \sigma(\alpha) \equiv \alpha \bmod w\}$$

The local inertia group sits inside the following canonical short exact sequence (there is also a global variant, but we shan't need that):

$$1 \to I_{L_w/K_v} \to \mathrm{Gal}(L_w/K_v) \to \mathrm{Gal}\left(\mathcal{O}_{L_w}/w\mathcal{O}_{L_w} \middle/ \mathcal{O}_{K_v}/v\mathcal{O}_{K_v}\right) \to 1 \tag{1}$$

Further recall that the inertia subgroup measures ramification of the prime v in both extensions; the main point of considering local extensions is that there is no other prime to consider there. The ramification behaviour of a global extension L/K is described by the collection of ramification behaviours of all local extensions L_w/K_v. In particular, L/K is unramified at a prime v iff L_w/K_v is unramified. ◦

For $v \notin S$ nonarchimedean, i_v is trivial on the inertia subgroup I_{K_v} of G_{K_v}. Indeed, an element of the local inertia subgroup $I_{K_v} \leq G_{K_v}$ is mapped to an element of the global inertia subgroup $I_{K,S,v} \leq G_{K,S}$, which is trivial since v is K_S^{al}/K is unramified at v by assumption. In other words, i_v descends to a homomorphism $i_v : G_{K_v}/I_{K_v} \to G_{K,S}$.

Let $k_v := \mathcal{O}_{K_v}/v\mathcal{O}_{K_v}$ denote the residue field of K_v, and $k_{\mathrm{al},v} := \mathcal{O}_{K_v^{\mathrm{al}}}/v\mathcal{O}_{K_v^{\mathrm{al}}}$. The sequence (1) gives us a canonical isomorphism $G_{K_v}/I_{K_v} \xrightarrow{\sim} G_{k_v} := \mathrm{Gal}(k_{\mathrm{al},v}/k_v)$. The latter group has a canonical topological generator, the Frobenius element $\varphi_v : x \mapsto x^{\# k_v}$. Therefore i_v is determined by its image on φ_v, which we denote by $\mathrm{Frob}_v \in G_{K,S}$, and which is, as noted above, unique up to conjugation.

For v archimedean, there are two cases. If v is complex, then $K_v \simeq \mathbb{C}$, which is algebraically closed, so $G_{K_v} = 1$, and i_v is the trivial map. If v is real, then $K_v \simeq \mathbb{R}$, in which case G_{K_v} is the group of order 2, so i_v is determined by its image on the nontrivial element, which we call, in analogy with the nonarchimedean case, the Frobenius.

To sum up, originally we wanted to study $G_{K,S}$, but seems opportune to study it together with the 'extra structure' given by the conjugacy classes of the i_v. By our discussion above, this is equivalent to studying $G_{K,S}$ together with conjugacy classes of Frobenii $\mathrm{Frob}_v \in G_{K,S}$ for places $v \notin S$, and the i_v's for all $v \in S$.

1.2 What is known without using Galois deformation theory?

Very roughly speaking, the structure of the global Galois group is complicated, while the local ones are much more docile. We don't need to say much more here. See [NSW20, VII§5 and IX] for more on this than anyone can handle.

The abelianisation $G_{K,S}^{\mathrm{ab}}$ is also well understood: this is exactly what class field theory does, describing abelian extensions of (local and global) fields with specific ramification properties. The problem is that this is far from the full picture, since $G_{K,S}$ is not abelian.

The birthplace of class field theory is the study of abelian extensions of $K = \mathbb{Q}$ via the Kronecker–Weber theorem. In this case, the discussion of the preceding subsection can be made more explicit, at least in the abelianisation. Namely, it is known that the maximal abelian extension of $\mathbb{Q}$ unramified outside S is $\mathbb{Q}(\mu_S)$ where μ_S is the group of roots of unity of order coprime to primes outside S. We have the following isomorphisms:

$$\begin{array}{ccccccc} G_{\mathbb{Q},S}^{\mathrm{ab}} & \simeq & \mathrm{Gal}(\mathbb{Q}(\mu_S)/\mathbb{Q}) & \simeq & \mathrm{Aut}(\mu_S) & \simeq & \prod_{p \in S} \mathbb{Z}_p^\times \\ \mathrm{Frob}_\ell & \longmapsto & & & & \longrightarrow & (\ldots, \ell, \ldots) \end{array}$$

1.3 Galois representations

But alas, we want to study the non-abelian group $G_{K,S}$. We also know that our study will be only up to conjugation. This, however, makes is natural to consider representations: those allow, by their very nature, only statements which are up to conjugation anyway. Therefore we want to study Galois representations $\rho : G_{K,S} \to \mathrm{GL}_N(A)$ for some topological ring A.

In this language, studying the package described in Section 1.1 means investigating such representations together with their restrictions to the local Galois groups. For $v \notin S$, this is described by $\mathrm{Tr}(\rho(\mathrm{Frob}_v))$ (which is well-defined). According to Mazur, knowing these traces is often enough to describe ρ up to equivalence. Using Chebotaryov's density theorem, we may not even need this at all $v \notin S$, but only a subset of density 1.

Remark 1.1. Class field theory can be seen as the study of one-dimensional representations of $G_{K,S}$. This may not be obvious from the usual points of view of standard class field theory courses. See [KKS12, p. 185] for an intuitive introduction. There is a more technical exposition in [Kre09, §5.4]. ◦

2 Universal deformation rings and number theory

We retain the notation of the previous sections. In particular, Π is a p-finite profinite group; keep in mind that later this will be set to be the absolute Galois group $G_{K,S}$ associated with some number field K and a set of places S.

In this section, we will relate the obstruction behaviour of a residual Galois representation for K to Euler characteristics and ultimately even to the Leopoldt conjecture.

Warning. In this section, there will be two different notions of dimension present: Krull dimension, denoted dim, and k-vector space dimension, denoted $\dim_k$.

2.1 The universal deformation ring of a representation

As in previous talks in this seminar, we will now study universal objects and the like. We start out by an existence statement, see [Maz97, §10] as well as [Maz89, §1.2].

Proposition 2.1 ([Maz97, §10, Proposition])**.** *If $\bar{\rho} : \Pi \to \mathrm{GL}_N(k)$ is absolutely irreducible then there are*

- *a coefficient-ring $R(\bar{\rho})$ with residue field k, called the* universal coefficient ring*, and*
- *a deformation $\rho^{\mathrm{univ}} : \Pi \to \mathrm{GL}_N(R(\bar{\rho}))$, called the* universal deformation*,*

such that for any coefficient-ring A with residue field k, and any deformation $\rho : \Pi \to \mathrm{GL}_N(A)$ of $\bar{\rho}$ to A there is a unique coefficient-ring morphism $h : R(\bar{\rho}) \to A$ (which, by definition, induces identity on residue fields) such that $\rho = \mathrm{GL}_N(h) \circ \rho^{\mathrm{univ}}$. In diagram:

$$\begin{array}{ccc} \Pi & \xrightarrow{\rho^{\mathrm{univ}}} & \mathrm{GL}_N(R(\bar{\rho})) \\ & {\scriptstyle \forall\rho}\searrow & \downarrow {\scriptstyle \exists!\, \mathrm{GL}_N(h)} \\ & & \mathrm{GL}_N(A) \end{array}$$

Remark 2.2. Universality can also be formulated in representability terms. Recall that $D_{\bar{\rho}}$ is the absolute deformation functor, assigning to a coefficient-ring A with residue field k the set of strict equivalence class of deformations of $\bar{\rho}$ to A. In the above case, this functor is representable by $R(\bar{\rho})$, that is, $D_{\bar{\rho}}(A) \simeq \mathrm{Hom}_{W(k)\text{-alg}}(R(\bar{\rho}), A)$ for all A, where $W(k)$ is the ring of Witt vectors. If you are unfamiliar with Witt vectors, all you need know here is that $W(k)$ is a ring with given residue field k, which is canonical and in some sense universal. ◦

Remark 2.3. The proof shall be omitted, but we still make some remarks. From the point of view of the seminar, it is natural to approach the statement via Schlessinger's theorem, which is the approach of [Maz89]. One should raise the question how absolute irreducibility is used, and whether we can still say something weaker for general $\bar{\rho}$. The answer to the first question is that to the verification of Schlessinger's criteria uses a result similar to Schur's lemma, which is only available for irreducible representations. If $\bar{\rho}$ is not absolutely irreducible, then a versal deformation exists. ◦

Remark 2.4. The universal deformation $(R(\bar{\rho}), \rho^{\mathrm{univ}})$ is uniquely determined up to twist-equivalence, i.e. up to tensoring by 1-dimensional representations. ◦

Example 2.5. In the case $N = 1$, we can give a somewhat explicit description of the universal deformation, as explained in [Maz89, §1.4]. Let $\bar{\rho} : \Pi \to k^\times$ be given. Then $\Lambda = R(\bar{\rho}) =$

$W(k)[\![\Pi^{\mathrm{ab},p}]\!]$ where $\Pi^{\mathrm{ab},p}$ is the p-completion of the the abelianisation of Π, and double brackets denote the completed group algebra. Let $\pi : \Pi \twoheadrightarrow \Pi^{\mathrm{ab},p}$ be the projection map, and let $\tilde{\rho} : \Pi \to W(k)$ be the Teichmüller lift. Then the universal deformation is

$$\rho^{\mathrm{univ}} : \Pi \to W(k)^\times \cdot \Pi^{\mathrm{ab},p} \hookrightarrow R(\bar{\rho})^\times, x \mapsto \tilde{\rho}(x) \cdot [\pi(x)]$$

where $[\pi(x)]$ denotes the image of the group element $\pi(x) \in \Pi^{\mathrm{ab},p}$ in the completed group ring. ◦

Recollection. A representation is absolutely irreducible if it remains irreducible under any extension of the ground field. In particular, by taking the trivial field extension, it is itself irreducible. See [CR81, (3.42)] for more on this. ◦

Remark. I am skipping two statements related to universal deformation rings of absolutely irreducible representations. One gives a description of the Zariski tangent module via Kähler differentials over a quotient of $R(\bar{\rho})$, see [Maz97, §23, p.290]. The other one gives an explicit description of the quotient of the universal deformation ring pro-representing 'determinant δ' deformations of an absolutely irreducible residual representation, see [Maz97, §24, p.292]. ◦

2.2 Adjoint representations and obstruction classes

Definition 2.6. Let $\bar{\rho} : \Pi \to \mathrm{GL}_N(k)$ be a residual representation. The *adjoint representation* $\mathrm{Ad}(\bar{\rho})$ has underlying space $M_N(k)$ with action via $\bar{\rho}$ and conjugation, that is, for $g \in \Pi$ and $m \in M_N(k)$, $g * m = \bar{\rho}(g)^{-1} m \bar{\rho}(g)$. ◦

Lemma 2.7. *Let Π be any group, and let $\bar{\rho} : \Pi \to \mathrm{GL}_N(k)$ be an absolutely irreducible residual representation. Then $H^0(\Pi, \mathrm{Ad}(\bar{\rho})) = k$. In other words, the only matrices commuting with every matrix in the image of $\bar{\rho}$ are the scalar matrices.*

Proof. Let $m \in M_N(k)$ be a matrix commuting with the image of $\bar{\rho}$. Let $\lambda \in k^{\mathrm{al}}$ be an eigenvalue of m in the algebraic closure, and let $v \in k^{\mathrm{al},N}$ be a corresponding nonzero eigenvector. Then for all $g \in \Pi$, the vector $\bar{\rho}(g)v \in k^{\mathrm{al},N}$ is also an eigenvector:

$$m\bar{\rho}(g)v = \bar{\rho}(g)mv = \bar{\rho}(g)\lambda v = \lambda\bar{\rho}(g)v$$

This means that for all $g \in \Pi$, $\bar{\rho}(g)$ descends to a map $E_\lambda \to E_\lambda$ where $E_\lambda \leq k^{\mathrm{al},N}$ denotes the eigenspace of λ. In other words, E_λ is a Π-submodule of $k^{\mathrm{al},N}$. But $\bar{\rho}$ is absolutely irreducible, and since $E_\lambda \neq 0$ (it contains v), it must be the whole $k^{\mathrm{al},N}$. Hence $m = \lambda \cdot 1_N$ is a scalar matrix, as claimed. □

Definition 2.8. Let $d^i := \dim_k H^i(\Pi, \mathrm{Ad}(\bar{\rho}))$ be the k-dimension of the group (later: Galois) cohomology groups of the adjoint representation, and write $\delta := d^1 - d^2$. ◦

Remark. Recall that we have seen in talks 5 and 7 that, loosely speaking, the space of deformations is an H^1-torsor, and obstructions come from H^2. So d^1 measures the size of the deformation space, and $d^2 = 0$ iff the deformation problem is unobstructed. ◦

Recall the notion of an obstruction class from talk 7. We briefly summarise what these are in the representation-theoretical setting. Let $A_1 \twoheadrightarrow A_0$ be a surjective morphism of complete noetherian local rings with residue field k (that is, $A_1 \twoheadrightarrow A_0$ is an epi in $\mathcal{C}_\Lambda(k)$). Let I be the kernel, and assume that $I \cdot \mathfrak{m}_{A_1} = 0$. Then there is a short exact sequence of k-vector spaces $0 \to I \to A_1 \to A_0 \to 0$.

Proposition 2.9. *Let $\rho_0 : \Pi \to \mathrm{GL}_N(A_0)$ be a deformation of $\bar{\rho} : \Pi \to \mathrm{GL}_N(k)$ to A_0. Then there exists a so-called* obstruction class *$\mathcal{O}(\rho_0) \in H^2(\Pi, I \otimes_k \mathrm{Ad}(\bar{\rho})) = H^2(\Pi, \mathrm{Ad}(\bar{\rho})) \otimes_k I$ depending* *only on ρ_0 that vanishes iff there is a deformation $\rho_1 : \Pi \to \mathrm{GL}_N(A_1)$ of ρ_0 (and thus of $\bar{\rho}$) to A_1.*

Proof. One writes down a so-called obstruction cocylce, and check that it does what it needs to do. The proof in [Maz89] contains a slight mistake. See Luca's notes for more on this. □

Warning. The notation $\mathcal{O}(\rho_0)$ suppresses the ring A_1.

2.3 Bounding the Krull dimension

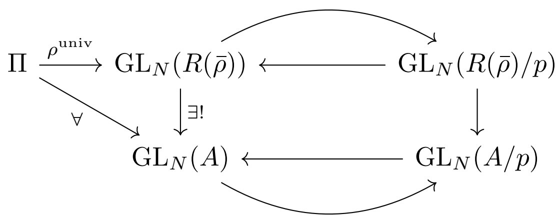
Proposition 2.10 ([Maz89, §1.6, Proposition 2])**.** *Let $\bar{\rho}$ be absolutely irreducible (so that it has a universal deformation ring $R(\bar{\rho})$). Then there is a lower bound for the Krull dimension of $R(\bar{\rho})/p$:*

$$\dim R(\bar{\rho})/pR(\bar{\rho}) \geq \delta = d^1 - d^2$$

Proof. Our strategy is to first construct a surjection $k[[T_1, \ldots, T_{d^1}]] \twoheadrightarrow R(\bar{\rho})/p$, which will in turn give rise to a morphism whose injectivity will be equivalent to the inequality above.

First recall that, as we have seen in the previous talk, $t_{D_{\bar{\rho}},k} \simeq H^1(\Pi, \mathrm{Ad}(\bar{\rho}))$.

Next, note that the mod p reduction $R(\bar{\rho})/p$ is the universal deformation ring for *characteristic* p deformations of $\bar{\rho}$. Indeed, this is clear from the following diagram:



Here A is runs through coefficient-rings, and the diagonal arrow runs through representations to them. The left vertical arrow is given by Proposition 2.1, which and this induces the right vertical arrow. The horizontal arrows in the square are liftings, and the bendy arrows are splittings to them.

Moreover, the Zariski tangent space $t_{D_{\bar{\rho}},k}$ can also be realised as the k-dual to the Zariski cotangent space $\mathfrak{m}_{R(\bar{\rho})}/\left(\mathfrak{m}^2_{R(\bar{\rho})}, p\right)$, and this agrees with the functorial definition. Now consider the local k-algebra $F := k[[T_1, \ldots, T_{d^1}]]$ defined as the ring of formal power series in d^1 variables. This has maximal ideal $\mathfrak{m}_F = T_1F + \ldots + T_{d^1}F$, its cotangent space is $\mathfrak{m}_F/\mathfrak{m}_F^2 = T_1k + \ldots + T_{d^1}k \simeq k^{\oplus d^1}$, and the Zariski tangent space, being the k-dual, is also isomorphic to $k^{\oplus d^1}$. In particular, it is isomorphic to $t_{D_{\bar{\rho}},k}$. This allows us to consider a continuous homomorphism $F \to R(\bar{\rho})/p$ that induces an isomorphism on Zariski tangent spaces.

This map is surjective by [Sch68, Lemma 1.1], which states that surjectivity of maps in $\mathcal{C}_\Lambda(k)$ can be checked on induced maps of Zariski cotangent spaces (the proof is unsurprising): here we even have an induced isomorphism of Zariski tangent spaces, whence the induced map of cotangent spaces will also be an iso, and in particular, surjective.

Let J denote the kernel. Then the exact sequence

$$0 \to J \to F \to R(\bar{\rho})/p \to 0 \tag{2}$$

gives rise to an exact sequence

$$0 \to J/\mathfrak{m}_F \cdot J \to F/\mathfrak{m}_F \cdot J \to R(\bar{\rho})/p \to 0 \tag{3}$$

Let $\rho_0 : \Pi \to \mathrm{GL}_N(R(\bar\rho)/p)$ be induced by ρ^{univ} (in other words, ρ_0 is the composition of ρ^{univ} with the top bendy arrow in the previous diagram). Consider the obstruction class associated with (3): that is, in the construction of Proposition 2.9, set $I := J/\mathfrak{m}_F J$, $A_1 := F/\mathfrak{m}_F J$, $A_0 := R(\bar\rho)/p$; note that the condition $I \cdot \mathfrak{m}_{A_1} = J/\mathfrak{m}_F J \cdot \mathfrak{m}_F/\mathfrak{m}_F J = 0$ is satisfied. We obtain the obstruction class $\mathcal{O}(\rho_0) \in H^2(\Pi, \mathrm{Ad}(\bar\rho)) \otimes_k J/\mathfrak{m}_F J$.

Consider the following map of k-vector spaces:

$$\begin{aligned} \mathrm{Hom}_k\left(J/\mathfrak{m}_F J, k\right) &\to H^2(\Pi, \mathrm{Ad}(\bar\rho)) \otimes_k k \simeq H^2(\Pi, \mathrm{Ad}(\bar\rho)) \\ f &\mapsto (1 \otimes f)(\mathcal{O}(\rho_0)) \end{aligned} \tag{4}$$

We claim that it is injective. ↯ Suppose $0 \neq f \in \mathrm{Hom}_k\left(J/\mathfrak{m}_F J, k\right)$ is in the kernel. Then

$$F/\mathfrak{m}_F J \Big/ \ker(f) \simeq \mathrm{im}(f) \simeq k \tag{5}$$

and (3) gives rise to a short exact sequence

$$0 \to F/\mathfrak{m}_F J \Big/ \ker(f) \to R' := F/\mathfrak{m}_F J \Big/ \ker(f) \to R(\bar\rho)/p \to 0 \tag{6}$$

Just like (3), this also gives rise to an obstruction class $\mathcal{O}_{R'}(\rho_0)$ which measures the obstruction of deforming R'. By the assumption on f, this class vanishes.

As noted above in this proof, the ring $R(\bar\rho)/p$ is the universal deformation ring of characteristic p deformations, and R' is of characteristic p (since it is a quotient of F). Therefore the vanishing of the obstruction class $\mathcal{O}_{R'}(\rho_0)$ implies that the sequence (6) must split. The ring R' inherits the property of F that $R' \to R(\bar\rho)/p$ induces an isomorphism of Zariski tangent spaces. However, as noted above in (5), the kernel is one-dimensional; since the (Krull) dimension of a ring is always less or equal to the dimension of the Zariski (co)tangent space, this means that the Zariski tangent space of the kernel has dimension ≥ 1. This is absurd. ↯

Injectivity of (4) yields the following inequality of k-dimensions:

$$\dim_k \left(\mathrm{Hom}_k(J/\mathfrak{m}_F J, k)\right) \leq \dim_k H^2(\Pi, \mathrm{Ad}(\bar\rho)) = d^2$$

Since the dual vector space has the same dimension as the original, it follows that

$$\dim_k \left(J/\mathfrak{m}_F J\right) \leq d^2$$

Using an appropriate version of Nakayama's lemma, it follows that J is generated by at most d^2 elements, which can be chosen to be lifts of some k-basis of $J/\mathfrak{m}_F J$. Returning to the short exact sequence (2), and comparing Krull dimensions of F and $R(\bar\rho)/p$, we have that each of these d^2 generators of J lowers the Krull dimension by at most one, see [Sta22, Lemma 00KW]. Hence

$$\dim R(\bar\rho)/p \geq \dim(F) - d^2 = d^1 - d^2 = \delta$$

□

Remark 2.11. From the proof we see that there is an upper bound $d^1 \geq \dim R(\bar\rho)/p$: the Zariski (co)tangent space has k-dimension d^1, which is always larger or equal to the Krull dimension for local rings. ∘

2.4 Euler–Poincaré characteristic and Leopoldt's conjecture

Let $n := (K : \mathbb{Q})$ be the degree of K, and assume that S contains all infinite places and all places of K above p. Write $G_v \subseteq G_{K,S}$ for the decomposition group of K_S^{al}/K at v, keeping in mind that we understand this to be of order 2 when v is real, and trivial when v is complex. Further assume that $\bar\rho$ is absolutely irreducible so that it makes sense to consider its universal deformation ring.

Corollary 2.12 ([Maz89, §1.10, Proposition 5]). *In the situation above, there is the following formula for $\delta = d^1 - d^2$:*

$$\delta = nN^2 + 1 - \sum_{v \in S_\infty} \dim_k H^0(G_v, \mathrm{Ad}(\bar{\rho}))$$

The proof will use the following:

Theorem 2.13 (Global Euler–Poincaré characteristic formula). *Let K be a number field of degree n. Assume that S contains all infinite places, and let M be a finite (as a set) $G_{K,S}$-module of order $\#M$ an S-unit (i.e. invertible away from S). Then*

$$\frac{\#H^0(G_{K,S}, M) \cdot \#H^2(G_{K,S}, M)}{\#H^1(G_{K,S}, M)} = \frac{\prod_{v \in S_\infty} \#H^0(G_v, M)}{(\#M)^n}$$

Proof. The proof lies beyond the scope of this talk. The reference given in [Maz97] is a ghost reference; an actual proof can be found in [NSW20, (8.7.4)]. The proof is highly nontrivial; among other things, the Poitou–Tate sequence is involved. On the left hand side, we have global Galois cohomology, while the right hand side consists of a collection of local Galois cohomology groups. The Poitou–Tate sequence is a 9-term exact sequence of exactly this type: it establishes a connection between global and local Galois cohomology, see [NSW20, (8.6.10)] for the precise statement. □

Proof (Proof of Corollary 2.12). The global Euler–Poincaré characteristic formula is applicable to $M := \mathrm{Ad}(\bar{\rho})$ because $\#M = (\#k)^{N^2}$ is a p-power, and by assumption all primes of K above p are in S, so $\#M$ is an S-unit.

Since $\bar{\rho}$ is absolutely irreducible, $H^0(G_{K,S}, \mathrm{Ad}(\bar{\rho}))$ is a 1-dimensional k-vector space by Lemma 2.7.

Now take the base $\#k$ logarithm of both sides of Theorem 2.13: on the left hand side, we have $1 + d^2 - d^1$, while on the right hand side $\sum_{v \in S_\infty} \#H^0(G_v, \mathrm{Ad}(\bar{\rho})) - nN^2$. The formula for $\delta = d^1 - d^2$ follows. □

2.4.1 Example: $N = 1$

Let $N := 1$. As usual, write r_1 resp. r_2 for the number of real resp. complex places of K. Recall that $n = r_1 + 2r_2$.

Claim 2.14. $\delta = r_2 + 1$.

Proof. By Corollary 2.12, we have the following formula for δ:

$$\delta = (r_1 + 2r_2) \cdot 1 + 1 - \sum_{v \in S_\infty} \dim_k H^0(G_v, \mathrm{Ad}(\bar{\rho}))$$

Since $N = 1$, the space $\mathrm{GL}_N(k)$ is commutative, and since $\mathrm{Ad}(\bar{\rho})$ is endowed with a group action via conjugation, this means that for all v, the group of invariants $H^0(G_v, \mathrm{Ad}(\bar{\rho}))$ is the whole space $\mathrm{Ad}(\bar{\rho}) = M_1(k) = k$ (as sets). Therefore the sum in the equation above is $\#S_\infty = r_1 + r_2$, and the claim follows. □

Recollection. We recall Leopoldt's conjecture. Fix a number field K and a rational prime p. The conjecture is known for $K/\mathbb{Q}$ abelian. This recollection is based on [Was97, §5.5].

One formulation of the conjecture is as follows: let

$$\sigma_1, \ldots, \sigma_{r_1}, \sigma_{r_1+1}, \overline{\sigma_{r_1+1}}, \ldots, \sigma_{r_1+r_2+1}, \overline{\sigma_{r_1+r_2+1}}$$

be the embeddings of K, and fix a basis $\varepsilon_1, \ldots, \varepsilon_{r_1+r_2-1} \in \mathcal{O}_K^\times$ for the units modulo roots of unity using Dirichlet's unit theorem. Write $\delta_i := 1$ if σ_i is real and $\delta_i := 2$ if it is complex. Define the p-adic regulator as $R_{K,p}(\varepsilon_1, \ldots, \varepsilon_{r_1+r_2-1}) := \det(\delta_i \log_p(\sigma_i \varepsilon_j))_{i,j}$; this is well-defined up to sign. Leopoldt's conjecture states $R_{K,p} \neq 0$.

There is another formulation relating the 1-units of the global field K to the 1-units of each of its completions $K_\mathfrak{p}$ at primes $\mathfrak{p}$ above p. This is more technical to write down, but ultimately more useful if we want to use class field theory (and we do!). For all $\mathfrak{p} \mid p$, write $\mathfrak{U}_\mathfrak{p}$ for the local units of $K_\mathfrak{p}$, and $\mathfrak{U}_\mathfrak{p}^{(1)}$ for the 1-units. Let $\mathfrak{U}_p := \prod_{\mathfrak{p}|p} \mathfrak{U}_\mathfrak{p}$ and $\mathfrak{U}_p^{(1)} := \prod_{\mathfrak{p}|p} \mathfrak{U}_\mathfrak{p}^{(1)}$. There is a diagonal embedding $\sigma : \mathcal{O}_K^\times \to \mathfrak{U}_p$. Consider the group $\mathfrak{U}_p^{(1)} \cap \overline{\sigma \mathcal{O}_K^\times}$, where overline denotes closure in the topology of $\mathfrak{U}_p$. The preimage of this group in $\mathcal{O}_K^\times$ is of finite index (the $N\mathfrak{p} - 1$th power maps to $\mathfrak{U}_\mathfrak{p}^{(1)}$), whence this preimage has $\mathbb{Z}$-rank equal to $\operatorname{rk}_\mathbb{Z} \mathcal{O}_K^\times = r_1 + r_2 - 1$. Leopoldt's conjecture states that the image $\mathfrak{U}_p^{(1)} \cap \overline{\sigma \mathcal{O}_K^\times}$ has $\mathbb{Z}_p$-rank $r_1 + r_2 - 1$ as well. ◦

Proposition 2.15. *Leopoldt's conjecture for K and p is equivalent to the equality $\dim R(\bar\rho)/p = \delta$ for one, and hence all, residual representations with $N = 1$. (All such representations are necessarily absolutely irreducible, so this makes sense.)*

Proof. By the description of the universal deformation ring for $N = 1$ in Example 2.5, we have

$$R(\bar\rho)/p \simeq k[\![G_{K,S}^{\mathrm{ab},p}]\!]$$

Here $G_{K,S}^{\mathrm{ab},p}$ means p-completion of the abelianisation of $G_{K,S}$. This group is naturally endowed with a $\mathbb{Z}_p$-module structure (the abelianisation is a $\mathbb{Z}$-module, hence the p-completion is a $\mathbb{Z}_p$-module).

Claim 2.16. *For any pro-p abelian group Γ, $\dim k[\![\Gamma]\!] = \dim_{\mathbb{Q}_p}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Gamma)$.*

Proof. First note that the right hand side is $\operatorname{rk}_{\mathbb{Z}_p} \Gamma$.

Assume $\Gamma \simeq \mathbb{Z}_p^r$; then the right hand side is r. To compute the left hand side, let $\mathcal{O}$ be the ring of integers in the unique unramified extension of $\mathbb{Q}_p$ with residue field k, and let $\pi \in \mathcal{O}$ be a uniformiser. Then it is a standard result in Iwasawa theory [NSW20, (5.3.5)] that there is a isomorphism of rings $\mathcal{O}[\![\Gamma]\!] \simeq \mathcal{O}[[T_1, \ldots, T_r]]$. Quotienting out by π on both sides, we obtain an iso $k[\![\Gamma]\!] \simeq k[[T_1, \ldots, T_r]]$. Then the dimension is easily seen to be r by using the formula $\dim R[T] = \dim(R) + 1$ inductively.

Now let Γ be an arbitrary pro-p abelian group of $\mathbb{Z}_p$-rank r. I don't know what to do now. □

Setting $\Gamma := G_{K,S}^{\mathrm{ab},p}$, it follows that $\dim k[\![G_{K,S}^{\mathrm{ab},p}]\!] = \dim_{\mathbb{Q}_p} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} G_{K,S}^{\mathrm{ab},p}$.

The equality $\dim_{\mathbb{Q}_p} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} G_{K,S}^{\mathrm{ab},p} = r_2 + 1$ is of course equivalent to $\operatorname{rk}_{\mathbb{Z}_p} G_{K,S}^{\mathrm{ab},p} = r_2 + 1$; we want to show that these are equivalent to Leopoldt's conjecture for K and p. This $\mathbb{Z}_p$-rank is, as shown during the proof of Corollary 1.3.5 in [For20] via global class field theory, $n - \operatorname{rk}_{\mathbb{Z}_p} \overline{\sigma \mathcal{O}_K^\times}$. (Warning: the notation of [For20] is slightly different from that of this script; definitions of the objects at hand are presented on page 18. A whole tower of fields is considered, instead of just one: our setup therefore corresponds to $n = 0$ there, and $G_{K,S}^{\mathrm{ab},p} = \mathfrak{X}_0$.) As $n = r_1 + 2r_2$, we get that $\operatorname{rk}_{\mathbb{Z}_p} G_{K,S}^{\mathrm{ab},p} = r_2 + 1$ iff $\operatorname{rk}_{\mathbb{Z}_p} \overline{\sigma \mathcal{O}_K^\times} = r_1 + r_2 - 1$. As we pointed out above, the preimage of $\mathfrak{U}_p^{(1)}$ under σ has coprime-to-p-index $\mathcal{O}_K^\times$ (at each $\mathfrak{p} \mid p$, the index divides $N\mathfrak{p} - 1$), wherefore $\operatorname{rk}_{\mathbb{Z}_p} \sigma \mathcal{O}_K^\times = \operatorname{rk}_{\mathbb{Z}_p} \mathfrak{U}_p^{(1)} \cap \overline{\sigma \mathcal{O}_K^\times}$. So $\operatorname{rk}_{\mathbb{Z}_p} G_{K,S}^{\mathrm{ab},p} = r_2 + 1$ iff $\operatorname{rk}_{\mathbb{Z}_p} \mathfrak{U}_p^{(1)} \cap \overline{\sigma \mathcal{O}_K^\times} = r_1 + r_2 - 1$, which is exactly Leopoldt's conjecture. □

Remark. Compare this with [Lan90, Theorem 5.2]. ◦

2.4.2 Example: $N = 2$

Let $N := 2$, $K := \mathbb{Q}$, $p > 2$, and consider a residual representation $\bar{\rho} : G_{\mathbb{Q},S} \to \mathrm{GL}_2(k)$.

We recall that for a number field, an automorphism is called a complex conjugation if under real embeddings, it becomes the identity, and under complex embeddings, it becomes an involution. For an element of some absolute Galois group, by a complex conjugation we mean an automorphism which becomes a complex conjugation in the previous sense when restricted to finite extensions.

Definition 2.17. We say that $\bar{\rho}$ is *even* if there exists a complex conjugation $\tau \in G_{\mathbb{Q},S}$ with image $\bar{\rho}(\tau) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. On the other hand, $\bar{\rho}$ is called *odd* if there exists a complex conjugation $\tau \in G_{\mathbb{Q},S}$ for which $\bar{\rho}(\tau) = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. ◦

Note that these are the only possible images a complex conjugation can have (it has eigenvalues ± 1). Further note that if the above conditions hold for one complex conjugation, then they hold for all complex conjugations.

Claim 2.18. *In this case,*

$$\dim(R(\bar{\rho})/p) \geq \begin{cases} 3 & \bar{\rho} \text{ is odd} \\ 1 & \bar{\rho} \text{ is even} \end{cases}$$

Proof. Follows directly from Corollary 2.12 and defintions. Indeed, $n = 1$, $N = 2$, and S_∞ consists of a single real place v. If $\bar{\rho}$ is even then both elements of G_v act trivially, so the space of invariants $H^0(G_v, \mathrm{Ad}(\bar{\rho}))$ is the whole $\mathrm{Ad}(\bar{\rho})$, which has dimension 4, and $\delta = 1$. If $\bar{\rho}$ is odd, then the nontrivial element (aka complex conjugation aka Frobenius) of G_v conjugates a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ into

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

so the space of invariants has dimension 2, and $\delta = 3$. □

3 Deformation conditions

The rest of this script concerns deformation conditions, which will be introduced in this section. The point of these is restricting the study of D_ρ to certain special representations. This is useful if one wishes to only consider deformations satisfying some specific condition, such as having fixed 'determinant' (as in Section 4) or being minimally ramified (Section 6). In Section 5, we will see how one can assemble a package of local deformation conditions into a global condition.

In order to do this, we need to set up an appropriate framework, which is why we make the following definitions:

Definition 3.1. Let $F_N(\Lambda; \Pi)$ be the category with

- objects (A', V') with A' artinian coefficient-Λ-algebras and V' free A'-modules of rank N with A'-linear continuous Π-action;
- morphisms $(A', V') \to (A'', V'')$ with $A' \to A''$ homomorphisms of artinian coefficient-Λ-algebras, $V' \to V''$ homomorphism of A'-modules such that these induce an isomorphism $V' \otimes_{A'} A'' \xrightarrow{\sim} V''$ compatible with the Π-action. ◦

Definition 3.2. Let $\mathcal{D}F_N \subseteq F_N(\Lambda;\Pi)$ be a full subcategory such that

1. If the source of an arrow of F_N is in $\mathcal{D}F_N$ then the target is also in $\mathcal{D}F_N$: if $((A',V') \to (A'',V'')) \in F_N$ and $(A',V') \in \mathcal{D}F_N$ then $(A'',V'') \in \mathcal{D}F_N$.

2. Let B, B', C be artinian coefficient-Λ-algebras with a cartesian diagram

$$\begin{array}{ccc} B \times_C B' & \xrightarrow{\beta\times_C 1} & B' \\ {\scriptstyle 1\times_C \beta'}\downarrow & \lrcorner & \downarrow{\scriptstyle \beta'} \\ B & \xrightarrow{\beta} & C \end{array}$$

   such that $(B \times_C B', V) \in F_N$. Let $V_B := V \otimes_{B\times_C B'} B$ resp. $V_{B'} := V \otimes_{B\times_C B'} B'$ where the $B\times_C B'$-module structures are given by the maps $1\times_C \beta'$ resp. $\beta\times_C 1$. Then $(B\times_C B', V) \in \mathcal{D}F_N$ iff $(B, V_B), (B', V_{B'}) \in \mathcal{D}F_N$.

3. $\mathcal{D}F_N$ is closed with respect to subobjects: if $((A',V') \to (A'',V'')) \in F_N$ and $A' \to A''$ is injective and $(A'',V'') \in \mathcal{D}F_N$ then $(A',V') \in \mathcal{D}F_N$.

A *deformation condition* $\mathcal{D}$ for $\bar{\rho}$ is a full subcategory $\mathcal{D}F_N$ as above containing $(k,\bar{V})$ as an object. The lift ρ is *of type* $\mathcal{D}$ if $(A, V(\rho)) \in \mathcal{D}F_N$. Its strict equivalence class is said to be *a* $\mathcal{D}$*-deformation of type* $\bar{\rho}$. ○

Remark. This definition is rather general: Π can be any p-finite profinite group. Later we will let Π be the absolute Galois group of a local field. Letting this local field run over certain completions of a global field, we will arrive at what is called a global Galois deformation problem. ○

Fix a deformation condition $\mathcal{D}$ for $\bar{\rho}$. In this section, we shall associate to this subfunctors $\mathcal{D}_{\bar{\rho}} \subset D_{\bar{\rho}}$ in the absolute deformation problem, and $\mathcal{D}_\rho \subset D_\rho$ in the relative deformation problem, and study the representability properties of these functors. More concretely, first we will introduce the notion of relative representability, and show that $\mathcal{D}_\rho$ is relatively representable. Afterwards, we will study nearly representability of $\mathcal{D}_\rho$ as well as pro-representability of $\mathcal{D}_{\bar{\rho}}$ via Schlessinger, as introduced in Luca's talk.

Recall the definitions of the categories $\mathcal{C}_\Lambda(A)$ resp. $\hat{\mathcal{C}}_\Lambda(A)$:

$$\hat{\mathcal{C}}_\Lambda(A) := \{\text{coefficient-}\Lambda\text{-algebras with a coefficient-}\Lambda\text{-algebra hom to } A\}$$
$$\mathcal{C}_\Lambda(A) := \text{full subcat of } \hat{\mathcal{C}}_\Lambda(A) \text{ with artinian objects}$$

Then $D_\rho : \hat{\mathcal{C}}_\Lambda(A) \to \mathsf{Set}$ is given letting $D_\rho(B)$ be the strict equivalence class of deformations of ρ to B.

Definition 3.3 (Relative case). First let $\mathcal{D}_\rho : \mathcal{C}_\Lambda(A) \to \mathsf{Set}$ is the functor given by

$$\mathcal{D}_\rho(B) := \left\{\xi \in D_\rho(B) \text{ representing } \rho_1 : \Pi \to \mathrm{GL}_N(B) \text{ relative } \rho : \Pi \to \mathrm{GL}_N(A) : \rho_1 \text{ is of type } \mathcal{D}\right\}$$

Extend this to a functor $\mathcal{D}_\rho : \hat{\mathcal{C}}_\Lambda(A) \to \mathsf{Set}$ by continuity, that is, $\mathcal{D}_\rho(B) := \varprojlim_n \mathcal{D}_\rho(B_n)$ where $B = \varprojlim_n B/\mathfrak{m}_B^n B$. (Note that $B/\mathfrak{m}_B^n B$ is always artinian.) ○

Remark. The $\mathcal{D}_\rho$ just defined is a subfunctor of D_ρ. Recall that a subfunctor $G \subseteq F : \mathcal{C} \to \mathsf{Set}$ is a functor such that for all $c \in \mathcal{C}$, $G(c) \subseteq F(c)$, and for all $(f : c \to c') \in \mathcal{C}$, $G(f) = F(f)|_{G(c)}$. ○

We make the analogous definition in the absolute case:

Definition 3.4 (Absolute case)**.** First let $\mathcal{D}_{\bar\rho} : \mathcal{C}_\Lambda(k) \to \mathsf{Set}$ is the functor given by

$$\mathcal{D}_{\bar\rho}(B) := \left\{\xi \in D_{\bar\rho}(B) \text{ representing } \rho_1 : \Pi \to \mathrm{GL}_N(B) \text{ relative } \bar\rho : \Pi \to \mathrm{GL}_N(k) : \rho_1 \text{ is of type } \mathcal{D}\right\}$$

Extend this to a functor $\mathcal{D}_\rho : \hat{\mathcal{C}}_\Lambda(k) \to \mathsf{Set}$ by continuity. ◦

3.1 Relative representability of $\mathcal{D}_\rho$

In the following, I will use F and $\mathcal{F}$ instead of D and $\mathcal{D}$. This is because I would like to reserve the notation $\mathcal{D}$ for deformation conditions, whereas here sometimes we will be talking about more general functors. This deviates from the notation of [Maz97].

Definition 3.5. Let $\mathcal{F} \subseteq F : \mathcal{C}_\Lambda(k) \to \mathsf{Set}$ be a functor and a subfunctor such that $\mathcal{F}(k) = F(k) = *$. Then $\mathcal{F} \subseteq F$ is *relatively representable* if for all diagrams

$$\begin{array}{ccc} & & B' \\ & & \downarrow{\scriptstyle\beta'} \\ B & \xrightarrow{\beta} & C \end{array}$$

in $\mathcal{C}_\Lambda(k)$, the square

$$\begin{array}{ccc} \mathcal{F}(B \times_C B') & \longrightarrow & \mathcal{F}(B) \times_{\mathcal{F}(C)} \mathcal{F}(B') \\ \downarrow & & \downarrow \\ F(B \times_C B') & \longrightarrow & F(B) \times_{F(C)} F(B') \end{array}$$

is cartesian (the horizontal maps are the ones induced by functoriality). ◦

From the definitions we automatically get the following:

Proposition 3.6 ([Maz97, §23, Proposition])**.** *If $\mathcal{D}$ is a deformation condition for $\bar\rho$ then $\mathcal{D}_\rho \subseteq D_\rho$ is relatively representable.*

Proof. The claim is that $\mathcal{D}_\rho(B \times_C B')$ is the fibre product of $D_\rho(B \times_C B')$ and $\mathcal{D}_\rho(B) \times_{\mathcal{D}_\rho(C)} \mathcal{D}_\rho(B')$ over $D_\rho(B) \times_{D_\rho(C)} D_\rho(B')$. This follows by first unraveling what a fibre product of sets is in case one of the maps is an embedding, then relating this to the definition of $\mathcal{D}_\rho$ by recalling the second property in the definition of deformation conditions. □

Remark. The following Lemma justifies the name ‘relatively representable’: roughly speaking, it shows that if $\mathcal{F}$ enjoys the same representability-related properties as F does.

I am not really satisfied with this piece of terminology, because to me, the word ‘relative’ conveys that the ambient functor F is somehow part of the data. But this is not the case: the Lemma shows that $\mathcal{F}$ has the same properties as F, and not some relative variant of them.

If I were to prove Proposition 3.8, then I would need the Lemma. However, I will not prove Proposition 3.8, as the proof is, as far as I can tell, not that interesting. ◦

Lemma 3.7. *If $\mathcal{F} \subseteq F$ is relatively representable, then the following hold.*

1. *If F satisfies H_i then so does $\mathcal{F}$.*
2. *If F satisfies T_k then so does $\mathcal{F}$.*
3. *If F is nearly representable then so is $\mathcal{F}$.*
4. *If F is representable then so is $\mathcal{F}$, and this is done by a quotient of the object representing F.*

3.2 Nearly representability of $\mathcal{D}_\rho$ and pro-representability of $\mathcal{D}_{\bar\rho}$

Recall the following definitions.

Definition ([Maz97, p. 277])**.** A covariant functor $D : \mathcal{C}_\Lambda(k) \to \mathsf{Set}$ with $D(k) = *$ is *pro-representable* if $F = \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, -)$ for some $R \in \hat{\mathcal{C}}_\Lambda$. (We have seen this in Talk 8.) ∘

Definition ([Maz97, p. 277])**.** A map $A \to C$ in $\mathcal{C}_\Lambda(k)$ is *small* if its kernel is principal and annihilated by the maximal ideal of A. ∘

Definition (absolute Zariski tangent space, [Maz97, p. 272])**.** Let $D : \mathcal{C}_\Lambda(k) \to \mathsf{Set}$ be a covariant functor with $D(k) = *$. Write $k[\varepsilon] := k \oplus \varepsilon \cdot k/(\varepsilon^2 = 0)$, and let $t_{D,k} := D(k[\varepsilon])$. We say that D satisfies *property T_k* if the natural map $D(k[\varepsilon] \times_k k[\varepsilon]) \to D(k[\varepsilon]) \times D(k[\varepsilon])$ is a bijection. In this case, $t_{D,k}$ can be endowed with a natural additive structure in the following way. First define the map $+$ by

$$\begin{aligned} k[\varepsilon] \times_k k[\varepsilon] &\to k[\varepsilon] \\ (x + y_1 \cdot \varepsilon, x + y_2 \cdot \varepsilon) &\mapsto x + (y_1 + y_2) \cdot \varepsilon \end{aligned}$$

Then define 'vector addition' on $t_{D,k}$ by the diagram

$$\begin{array}{ccccc} t_{D,k} \times t_{D,k} & \longrightarrow & & \longrightarrow & t_{D,k} \\ \| & & & & \| \\ D(k[\varepsilon]) \times D(k[\varepsilon]) & \xrightarrow{\simeq} & D(k[\varepsilon] \times_k k[\varepsilon]) & \xrightarrow{D(+)} & D(k[\varepsilon]) \end{array}$$

where the bottom left horizontal arrow is the inverse of the natural map in property T_k. There is also a multiplicative action of k on $t_{D,k}$, induced by the map $a * (x + y \cdot \varepsilon) := x + a.y \cdot \varepsilon$. This is compatible with the additive action, and defines a k-vector space structure on $t_{D,k}$. ∘

Definition (relative Zariski tangent module, [Maz97, p. 273])**.** Similarly, let $A[\varepsilon] := A \oplus \varepsilon \cdot A/(\varepsilon^2 = 0)$. We say that D satisfies *property T_A* if the natural map $D(A[\varepsilon] \times_A A[\varepsilon]) \to D(A[\varepsilon]) \times D(A[\varepsilon])$ is a bijection. As above, this allows one to endow $t_{D,A} := D(A[\varepsilon])$ with an A-module structure. ∘

Theorem (Schlessinger, [Maz97, p. 277])**.** *Let $D : \mathcal{C}_\Lambda(k) \to \mathsf{Set}$ be a covariant functor with $D(k) = *$. Then D is pro-representable iff for all diagrams*

$$\begin{array}{ccc} & & B' \\ & & \downarrow{\scriptstyle \beta'} \\ B & \xrightarrow{\beta} & C \end{array}$$

the following four conditions all hold:

H_1) If $B \to C$ is small then the natural map $D(B \times_C B') \to D(B) \times_{D(C)} D(B')$ is surjective.

H_2) If $B \to C$ is $k[\varepsilon] \to k$ then the natural map $D(B \times_C B') \to D(B) \times_{D(C)} D(B')$ is bijective.

H_3) T_k holds and $\dim_k t_{D,k}$ is finite.

H_4) If $\beta = \beta'$ and small then the natural map $D(B \times_C B) \to D(B) \times_{D(C)} D(B)$ is bijective.

We consider the following two weakenings of representability. The first one, having a pro-representable hull, is essentially the same as we have seen in talk 10.

Definition ([Maz97, p. 278])**.** Let $D : \mathcal{C}_\Lambda(k) \to \mathsf{Set}$ be a covariant functor with $D(k) = *$. A *pro-representable hull* for D is a pair (R, ε) where R is a coefficient-Λ-algebra, $D_R = \mathrm{Hom}_{\hat{\mathcal{C}}_\Lambda}(R, -)$ is the functor pro-represented by R, and $\varepsilon : D_R \to R$ is a smooth morphism of functors, such that the induced map between Zariski tangent k-vector spaces

$$t_{D_R,k} = \mathrm{Hom}_k\left(\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_\Lambda R), k\right) \to t_{D,k}$$

is an isomorphism.

Here smooth means the same as versal, and the condition on Zariski tangent spaces is the same as miniversal in talk 9. More explicitly, ε being smooth means that the strong surjectivity holds, i.e. for any surjection $B \twoheadrightarrow A$ in $\mathcal{C}_\Lambda$, the natural map $D_R(B) \to D_R(A) \times_{D(A)} D(B)$ is also surjective (see the notes to talk 9). ◦

Definition ([Maz97, p. 277])**.** Let $D : \mathcal{C}_\Lambda(A) \to \mathsf{Set}$ be a covariant functor with $D(A) = *$. Then D is *nearly representable* if it satisfies T_A and $t_{D,A}$ is finitely generated over A. ◦

Remark. In geometric terms, one may think about the Zariski cotangent space as the linear parts of functions vanishing at a certain point, while the tangent space is obtained as the dual of this. It can be shown that this is equivalent to the definition in the absolute case above (this is not obvious, see [Maz97, pp. 271–272]). Then pro-representability means that one may 'converge' to the deformation functor by representable functors, and this 'convergence' respects Zariski tangent spaces. Nearly representability is even weaker: then we only have information on the tangent module. The reason we care about these weaker notions is that our deformation functors will often fail to be representable, but will often satisfy many of these weaker conditions. ◦

Proposition 3.8 ([Maz97, §23, Corollary])**.** *The following hold:*

1. *If $\mathcal{D}$ is a deformation condition for $\bar\rho$ then $\mathcal{D}_{\bar\rho}$ satisfies H_1, H_2 and H_3.*
2. *$\mathcal{D}_{\bar\rho}$ has a pro-representable hull.*
3. *$\mathcal{D}_\rho$ is nearly representable.*
4. *If $\bar\rho$ is absolutely irreducible then $\mathcal{D}_{\bar\rho}$ is pro-representable, and this is done by a quotient of the ring pro-representing $\mathcal{D}_{\bar\rho}$.*

Proof. Omitted. The proof is supposed to be 'straightforward' by using Lemma 3.7 and Proposition 3.6. □

3.3 Tangent module to $\mathcal{D}_\rho$

According to the Corollary, $\mathcal{D}_\rho$ is nearly representable, which in particular means that $t_{\mathcal{D}_\rho,A} \doteq \mathcal{D}_\rho(A[\varepsilon])$ carries an A-module structure. Since $\mathcal{D}_\rho$ is by definition a subfunctor of D_ρ, the module $t_{\mathcal{D}_\rho,A}$ will be an A-submodule of $t_{D_\rho,A} \doteq D_\rho(A[\varepsilon])$.

In Luca's talk we will have seen a cohomological description for this module: $t_{D_\rho,A} \simeq H^1(\Pi, \mathrm{End}_A(V(\rho)))$, see [Maz97, §21, Proposition 1].

Definition 3.9. Let $H^1_{\mathcal{D}}(\Pi, \mathrm{End}_A(V(\rho)))$ denote the image of $t_{\mathcal{D}_\rho,A}$ under the isomorphism $t_{D_\rho,A} \xrightarrow{\sim} H^1(\Pi, \mathrm{End}_A(V(\rho)))$. In diagram:

$$\begin{array}{ccc} t_{\mathcal{D}_\rho,A} & \hookrightarrow & t_{D_\rho,A} \\ \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\ H^1_{\mathcal{D}}(\Pi, \mathrm{End}_A(V(\rho))) & \hookrightarrow & H^1(\Pi, \mathrm{End}_A(V(\rho))) \end{array}$$

Note that at this point, this is just a notation. $\circ$

4 Determinant conditions

Recall the definition of a determinant via exterior powers. For a vector space W of dimension N, $\bigwedge^i W$ is the universal object for alternating multilinear functions out of W^i. By functoriality, any $\varphi \in \operatorname{End} W$ gives rise to $\bigwedge^i \varphi \in \operatorname{End} \bigwedge^i W$. In the top dimension N, $\bigwedge^N W$ is one-dimensional, hence $\bigwedge^N \varphi$ acts by a scalar multiplication, and this scalar is the determinant of φ.

Fix a continuous homomorphism $\delta : \Pi \to \Lambda^\times$.

Definition 4.1. We say that $V(\rho)$ *has 'determinant δ'* if the action of Π on $\bigwedge^N V(\rho)$ is given via $\delta_A : \Pi \xrightarrow{\delta} \Lambda^\times \to A^\times$ where the last arrow comes from the Λ-module structure of A. We will write $F_N(\Lambda; \Pi; \delta)$ for the full subcategory of $F_N(\Lambda; \Pi)$ of modules of 'determinant δ'. $\circ$

Remark 4.2. The quotation marks are supposed to help us keep in mind the fact that the actual determinant of the representation $V(\rho)$ takes values in A, not in Λ. Further note that if $\rho' : \Pi \to \mathrm{GL}_N(A')$ is a deformation of $\rho : \Pi \to \mathrm{GL}_N(A)$ and if ρ' has "determinant δ" then so does ρ: indeed, write $h : A' \to A$ for the coefficient-Λ-homomorphism inducing the deformation, and then $\delta_A = h \circ \delta_{A'}$. $\circ$

Proposition 4.3 ([Maz97, §24, Proposition])**.** *Let $\mathcal{D}$ be the condition being given by having "determinant δ". Then $\mathcal{D}$ is a deformation condition. Moreover, there is a description of the group $H^1_{\mathcal{D}}(\Pi, \operatorname{End}_A(V(\rho)))$: let $\operatorname{End}^0_A(V(\rho)) \subseteq \operatorname{End}_A(V(\rho))$ denote the submodule of endomorphisms of trace zero; this is stable under the Π-action, and*

$$H^1_{\mathcal{D}}(\Pi, \operatorname{End}_A(V(\rho))) = H^1(\Pi, \operatorname{End}^0_A(V(\rho))) \subseteq H^1(\Pi, \operatorname{End}_A(V(\rho)))$$

Proof. The assertion about $\mathcal{D}$ being a deformation condition means that it satisfies the three numbered conditions in Definition 3.2. According to Mazur, this is 'straightforward to show'—which is indeed the case once one manages to get past the many, many definitions in his paper. To illustrate this, we show that the first condition holds: let $\psi : (A', V') \to (A'', V'')$ be a morphism in F_N, and assume that (A', V') is in $\mathcal{D}F_N$. Recall that A' resp. A'' are coefficient-Λ-algebras, and that V' resp. V'' are free modules over them of rank N with continuous linear Π-action; ψ being a morphism means what it should mean. Then by assumption, V' has 'determinant δ', i.e. the action of Π on $\bigwedge^N V'$ happens via $\delta_{A'} : \Pi \xrightarrow{\delta} \Lambda^\times \to A'^{,\times}$. Then since ψ is a morphism with all the nice compatibility properties one expects, the Π-action on $\bigwedge^N V''$ is via

$$\delta_{A''} = \psi|_{A'^{,\times}} \circ \delta_{A'} : \Pi \xrightarrow{\delta} \Lambda^\times \to A'^{,\times} \to A''^{,\times}$$

This means that V'' has 'determinant δ', hence (A'', V'') is in $\mathcal{D}F_N$, as desired. Checking the other two conditions should be doable in a similar fashion.

Claim 4.4. *There is a commutative diagram*

$$\begin{CD} 1 + \varepsilon \cdot M_N(A) @>{j}>> M_N(A) \\ @V{\det}VV @VV{\mathrm{Tr}}V \\ 1 + \varepsilon \cdot A @>{j}>> A \end{CD} \tag{7}$$

where the top resp. bottom horizontal maps are $j(1 + \varepsilon \cdot X) = X$ resp. $j(1 + \varepsilon \cdot x) = x$ for $X \in M_N(A)$ resp. $x \in A$.

Proof. Commutativity can be shown as follows. For $N = 2$, consider the matrix $1 + \varepsilon \cdot X \in 1 + \varepsilon \cdot M_N(A)$ in the top left corner of the diagram, where $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\mathrm{Tr}(j(1+\varepsilon \cdot X)) = \mathrm{Tr}\, X = a + d$, and $j(\det(1 + \varepsilon X)) = j(1 + \varepsilon a + \varepsilon d + \varepsilon^2 \cdot (ad - bc)) = a + d$, which indeed agree. For $N \geq 3$, the first computation holds without any modification. For the other one, proceed by induction: write the matrix as

$$1 + \varepsilon \cdot X = \begin{pmatrix} 1 + \varepsilon a & b \\ c & 1 + \varepsilon d \end{pmatrix}$$

where b resp. c are $1 \times (N-1)$ resp. $(N-1) \times 1$ vectors, and d is an $(N-1) \times (N-1)$ square matrix. Expand with respect to the first row. Elements of b are always multiplied by a determinant whose Laplacian expansion contains in each term ε with an exponent of at least two, wherefore these do not contribute. What remains is $(1+\varepsilon a) \cdot \det(1+\varepsilon d) = 1 + \varepsilon a + \varepsilon \,\mathrm{Tr}\, d + \varepsilon^2 (a \cdot \mathrm{Tr}\, d) = 1 + \varepsilon \cdot \mathrm{Tr}\, X$, which is sent to $\mathrm{Tr}\, X$ by j. Hence the claim. □

For $\rho : \Pi \to \mathrm{GL}_N(A)$ as above, let $\rho_0 : \Pi \to \mathrm{GL}_N(A) \hookrightarrow \mathrm{GL}_N(A[\varepsilon])$ be the composition of ρ with the inclusion induced by the inclusion $A \hookrightarrow A[\varepsilon]$. Let ρ' be a lifting of ρ to $A[\varepsilon]$, that is, $\rho' : \Pi \to \mathrm{GL}_N(A[\varepsilon])$ such that $h \circ \rho' = \rho$ where $h : \mathrm{GL}_N(A[\varepsilon]) \to \mathrm{GL}_N(A)$ is induced by $A[\varepsilon] \to A, \varepsilon \mapsto 0$.

Then $\rho'(g) \cdot \rho_0(g)^{-1} \in 1 + \varepsilon \cdot M_N(A)$: indeed, $h(\rho'(g) \cdot \rho_0(g)^{-1}) = h(\rho'(g)) \cdot h(\rho_0(g))^{-1} = \rho(g) \cdot \rho(g)^{-1} = 1$. Therefore it makes sense to consider the *difference cocycle*, which we will have encountered in Luca's talk:

$$\begin{aligned} c_{\rho'} : \Pi &\to M_N(A) \\ g &\mapsto j(\rho'(g) \cdot \rho_0(g)^{-1}) \end{aligned}$$

Commutativity of (7) implies that

$$\mathrm{Tr}(c_{\rho'}(g)) = j\big(\det(\rho'(g)) \cdot \det(\rho(g))^{-1}\big)$$

Therefore the cocycle $c_{\rho'}(-)$ has image in $M_N(A)^0$ iff the right hand side here is zero, which by definition of j is equivalent to $\det(\rho'(g)) \cdot \det(\rho(g))^{-1} = 1$, that is, $\det(\rho'(g)) = \det(\rho(g))$ for all $g \in \Pi$. Since ρ has 'determinant δ' by assumption, this means that $c_{\rho'}(-)$ takes values in $M_N(A)^0$ iff ρ' has 'determinant δ' as well.

The assertion $H^1_{\mathcal{D}}(\Pi, \mathrm{End}_A(V(\rho))) = H^1(\Pi, \mathrm{End}^0_A(V(\rho)))$ now follows from the identification of $t_{D_\rho, A} \simeq H^1(\Pi, \mathrm{End}_A(V(\rho)))$ in Luca's talk. Hopefully this will require no further explanation after just having seen the proof in the morning. □

5 Galois deformation conditions

Reference: [Maz97, §26].

Let K be a number field, and let Λ, p, and N be as above. Let S be a finite set of finite primes of K containing all primes above p.

Definition 5.1. A global Galois deformation problem for Λ, N, K, S is the collection of deformation conditions for $\Pi := G_{K_v}$ for all $v \in S$. ◦

Recall that by Definition 3.2, this means having for all $v \in S$ a full subcategory $\mathcal{D}_v F_N(\Lambda; G_{K_v})$ of $F_N(\Lambda; G_{K_v})$. Further recall that we have continuous homomorphisms $i_v : G_{K_v} \to G_{K,S}$. These give rise to 'restriction morphisms' $i_v : F_N(\Lambda; G_{K,S}) \to F_N(\Lambda; G_{K_v})$ (abuse of notation).

Definition 5.2. Given a global Galois deformation problem as above, let $\mathcal{D}F_N(\Lambda; G_{K,S})$ be the full subcategory of $F_N(\Lambda; G_{K,S})$ with objects

$$(A, V) \in \mathcal{D}F_N(\Lambda; G_{K,S}) \iff \forall v \in S : i_v((A, V)) \in \mathcal{D}_v F_N(\Lambda; G_{K_v}) \qquad \circ$$

We will also consider a 'fixed determinant' version of this notion. Let $\delta : G_{K,S} \to \Lambda^\times$ be a character.

Definition 5.3. Let $\mathcal{D}F_N(\Lambda; G_{K,S}; \delta)$ be the full subcategory of $\mathcal{D}F_N(\Lambda; G_{K,S})$ with objects of 'determinant δ. The *global Galois deformation problem with fixed determinant* is the corresponding deformation condition. $\circ$

Those who are not (yet) lost in this jungle of definitions should complain at this point: why is it valid to call $\mathcal{D}F_N(\Lambda; G_{K,S})$ and $\mathcal{D}F_N(\Lambda; G_{K,S}; \delta)$ deformation conditions? We only know that they are full subcategories of $\mathcal{D}F_N(\Lambda; G_{K,S})$ coming from some local deformation conditions. But of course things work out fine, as witnessed by the following:

Proposition 5.4 ([Maz97, §26, Proposition 1])**.** *Both*

$$\mathcal{D}F_N(\Lambda; G_{K,S}) \subseteq F_N(\Lambda; G_{K,S}) \text{ and } \mathcal{D}F_N(\Lambda; G_{K,S}; \delta) \subseteq F_N(\Lambda; G_{K,S}; \delta)$$

are full subcategories satisfying the two conditions in Definition 3.2.

Proof. This is supposed to be straightforward using the definitions and the fact that having fixed 'determinant δ' is a deformation condition (by Proposition 4.3). I shall not present the proof, as there are more interesting things to cover. $\square$

In the rest of §26, Mazur gives a description of the associated Zariski tangent modules in terms of $H^1_{\mathcal{D}_v}$-cohomology.

6 A local deformation problem: minimal ramification

In this section, we are going to see an example of a local Galois deformation condition. This appears to be (almost) the same as used by Wiles in his proof [Wil95, p. 458] of Fermat's last theorem, which already demonstrates how useful Galois deformations are.

Let k be, as before, a finite field of characteristic p. Let F be a local field of characteristic $\ell \neq p$ with Galois group G_F. Let $I \subset G_F$ be the inertia subgroup. We shall not to concern ourselves with the question of where F comes from; in Wiles's paper, it arises as a completion from the extension $\mathbb{Q}_S$ where $\mathbb{Q}_S/\mathbb{Q}$ is the maximal S-ramified extension of $\mathbb{Q}$.

Let $\bar{\rho} : G_F \to \mathrm{GL}_2(k)$ be a residual representation. In what follows, we shall treat two cases, labelled (A) and (B), simultaneously.

Definition 6.1. We say that we are in *case (A)* if $\bar{\rho}(I) \neq 1$ and $\bar{\rho}(I)$ is contained in a subgroup of $\mathrm{GL}_2(k)$ conjugate to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

Similarly, we say that we are in *case (B)* if $\bar{\rho}(I) \neq 1$ and $\bar{\rho}(I)$ is contained in a subgroup of $\mathrm{GL}_2(k)$ conjugate to $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$. $\circ$

Definition 6.2. Let ρ be a deformation of $\bar{\rho}$ to a coefficient-ring A with residue field k. Then ρ is *minimally ramified* if either (A) or (B) holds, and

(A) $\rho(I)$ is contained in a subgroup of $\mathrm{GL}_2(A)$ conjugate to $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

(B) $\rho(I)$ is finite and of order prime to p. ○

Remark. Using general facts about G_F, the intertia subgroup I as well as its tame quotient (the quotient of I by wild inertia), there are some alternative formulations for the notions just introduced. These prove useful in verifying Proposition 6.3. ○

Let k, Λ, and A be as above. We now define a condition $\mathcal{D}$ on local Galois representations: let $\mathcal{D}$ be the condition on a representation $\rho : G_F \to \mathrm{GL}_2(A)$ that its residual representation $\bar{\rho} : G_F \to \mathrm{GL}_2(k)$ fits either (A) or (B) and ρ satisfies the corresponding minimal ramification condition.

Proposition 6.3. *The condition $\mathcal{D}$ defined above is a deformation condition, and the cohomology group defined in Definition 3.9 is*

$$H^1_{\mathcal{D}}(G_F, \mathrm{End}_A(V)) = H^1(G_f, \mathrm{End}_A(V)^I) \subseteq H^1(G_F, \mathrm{End}_A(V))$$

where f is the residue field of F (finite field of characteristic ℓ).

Proof. We omit the proof. The first assertion requires checking that the conditions of Definition 3.2 hold: this is non-trivial commutative algebra combined with the alternative formulations alluded to above. The second assertion is, like Proposition 4.3, proven using the difference cocycle. □

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