RTG 2553:
Symmetries and classifying spaces:
Analytic, arithmetic and derived

UNIVERSITÄT DUISBURG ESSEN

Open-Minded

An equivariant p-adic Artin conjecture

Dr. Bence Forrás

Doctoral Researcher (10/2020–09/2023),
Funding: RTG
Master: Rheinische Friedrich-Wilhelms-Universität Bonn

1. Advisor: Prof. Dr. Nickel,
2. Advisor: Prof. Dr. Bertolini.

Essen Seminar for Algebraic Geometry and Arithmetic

Motivation

Theorem (Greenberg 1983)**.** Let $\chi : \mathcal{G} \to \mathbb{C}_p$ be a character with open kernel. There exists a fraction $G_{\chi,S}(T) \in \mathrm{Quot}(\mathcal{O}_{\mathbb{Q}_p(\chi)}[[T]])$ of power series such that

$$L_{p,S}(1-s,\chi) = G_{\chi,S}(u^s-1)/(\text{explicit polynomial})$$

where $\mathcal{O}_{\mathbb{Q}_p(\chi)}[\![\Gamma]\!] \simeq \mathcal{O}_{\mathbb{Q}_p(\chi)}[[T]]$, $\gamma \mapsto 1+T$, and u is the image of γ under the infinite part of the p-adic cyclotomic character.

Theorem (p-adic Artin conjecture, Wiles 1990)**.** $G_{\chi,S}(T) \in \mathbb{Z}_p[[T]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p^c$.

Theorem (refined p-adic Artin conjecture, Ritter–Weiss 2004)**.** $G_{\chi,S}(T) \in \mathcal{O}_{\mathbb{Q}_p(\chi)}[[T]]$.

Ritter–Weiss defined an equivariant element $\Phi_S \in \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times$ which specialises to the respective characacterwise p-adic Artin L-functions.

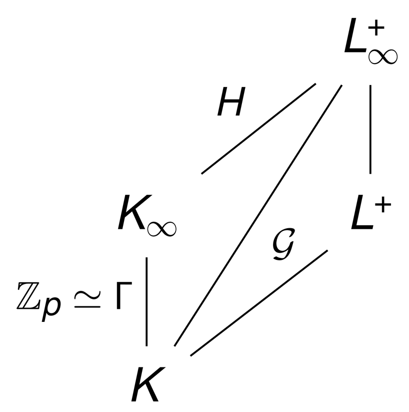
Goal. Introduce an *equivariant* version of the refined p-adic Artin conjecture, and prove it for certain classes of extensions.

Due to the trivial character, denominators are present in the definition of Φ_S. These are made more amenable by introducing a *non-empty smoothing set* T.

Nichifor–Palvannan [1] studied the question without smoothing, working away from the trivial character, in the case when $\mathcal{G} \simeq H \times \Gamma$ is a direct product cut out by an $\mathbb{F}_p^\times$-valued character.

Notation

- p denotes an odd rational prime
- L/K is a finite CM field extension such that $\zeta_p \in L$
- L^+ is the maximal totally real subfield of L
- K_∞/K is the cyclotomic $\mathbb{Z}_p$-extension, $L_\infty^+ := K_\infty L^+$



- $\mathcal{G} := \mathrm{Gal}(L_\infty^+/K) \simeq H \rtimes \Gamma$, $\gamma \in \Gamma$ topological generator
- Γ_0 nontrivial subgroup of Γ with central image in $\mathcal{G}$
- $\Lambda(\mathcal{G}) := \mathbb{Z}_p[\![\mathcal{G}]\!]$, $\mathcal{Q}(\mathcal{G}) := \mathrm{Quot}(\Lambda(\mathcal{G}))$
- $\mathfrak{z}(-)$ is the centre of a ring
- S, T are finite disjoint sets of places of K satisfying $S \supseteq S_{\mathrm{ram}}(L_\infty/K)$ and a technical condition on T
- $\mathbb{Q}_p(\chi) := \mathbb{Q}_p(\chi(g) : g \in \mathcal{G})$ for a character χ of $\mathcal{G}$
- $\mathbb{Q}_{p,\chi} := \mathbb{Q}_p(\chi(h) : h \in H)$ for a character χ of $\mathcal{G}$

Main Result

Consider the following equivariant p-adic Artin L-function attached to L_∞^+/K:

$$\Phi_S^T := \Phi_S \cdot \prod_{v \in T} \mathrm{nr}\left(1 - \varphi_{w_\infty}\right)$$

where $w_\infty \mid v$ is a fixed place of L_∞^+, nr means reduced norm, φ_{w_∞} is the Frobenius.

Conjecture (equivariant p-adic Artin conjecture)**.** Let $\mathfrak{M}$ be a $\Lambda(\Gamma_0)$-order in $\mathcal{Q}(\mathcal{G})$ containing $\Lambda(\mathcal{G})$. Then the smoothed equivariant p-adic Artin L-function Φ_S^T is in the image of the composite map

$$\mathfrak{M} \cap \mathcal{Q}(\mathcal{G})^\times \to K_1(\mathcal{Q}(\mathcal{G})) \xrightarrow{\mathrm{nr}} \mathfrak{z}(\mathcal{Q}(\mathcal{G}))^\times$$

where the first arrow is the natural map $x \mapsto [(x)]$.

Theorem 1. Suppose that for all irreducible characters χ with open kernel and all irreducible constituents $\eta \mid \mathrm{res}_H^{\mathcal{G}} \chi$, the extension $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified. Assume the equivariant Iwasawa main conjecture. Then the equivariant p-adic Artin conjecture *holds for all maximal $\Lambda(\Gamma_0)$-orders* in $\mathcal{Q}(\mathcal{G})$.

The total ramification condition above is satisfied, for instance, in the following cases:

- if $\mathcal{G} \simeq H \times \Gamma$ is a direct product;
- if $\mathcal{G}$ is a pro-p group;
- if for all prime divisors $q \mid \#H$, we have $p \nmid q-1$.

Using a result of Johnston–Nickel [2], we have a stronger and unconditional statement in the following special case:

Theorem 2. If p does not divide the order of the commutator subgroup of $\mathcal{G}$, then $\Phi_S^T \in \mathfrak{z}(\Lambda(\mathcal{G}))$, and the equivariant p-adic Artin conjecture holds for $\mathfrak{M} = \Lambda(\mathcal{G})$.

Sketch of proof

A major part of the proof of Theorem 1 is determining the *Wedderburn decomposition of $\mathcal{Q}(\mathcal{G})$*:

$$\mathcal{Q}(\mathcal{G}) \simeq \prod_{\chi \in \mathrm{Irr}(\mathcal{G})/\sim} M_{n_\chi}(D_\chi)$$

- χ runs through equivalence classes of irreducible characters of $\mathcal{G}$ with open kernel
- $\chi \sim \chi' \iff \exists \sigma \in \mathrm{Gal}(\mathbb{Q}_{p,\chi}/\mathbb{Q}_p) : \mathrm{res}_H^{\mathcal{G}} \chi = {}^\sigma(\mathrm{res}_H^{\mathcal{G}} \chi')$
- D_χ is a skew field, which is *a priori* poorly understood

In contrast to this, the decomposition of the group ring $\mathbb{Q}_p[H]$ is well understood:

$$\mathbb{Q}_p[H] \simeq \prod_{\eta \in \mathrm{Irr}(H)/\sim} M_{n_\eta}(D_\eta)$$

- η runs through equivalence classes of irreducible characters of H
- $\eta \sim \eta' \iff \exists \sigma \in \mathrm{Gal}(\mathbb{Q}_p(\eta)/\mathbb{Q}_p) : \eta = {}^\sigma(\eta')$
- D_η is a skew field with centre $\mathbb{Q}_p(\eta)$, determined by its Hasse invariant
- D_η has a unique maximal $\mathbb{Z}_p$-order $\mathcal{O}_{D_\eta}$

Fix χ, and let $\eta \mid \mathrm{res}_H^{\mathcal{G}} \chi$ be an irreducible constituent. The extension $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is cyclic, and there is a unique generator τ that acts as a power of γ on η.

Theorem 3. If $\mathbb{Q}_p(\eta)/\mathbb{Q}_{p,\chi}$ is totally ramified, then $D_\chi \simeq \mathrm{Quot}\left(\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]]\right)$.

The skew power series ring $\mathcal{O}_{D_\eta}[[X;\tau,\tau-\mathrm{id}]]$

- has underlying additive group $\mathcal{O}_{D_\eta}[[X]]$
- has multiplication rule $dX = \tau(d)X + (\tau-\mathrm{id})(d)$ for $d \in \mathcal{O}_{D_\eta}$
- possesses a Weierstrass theory due to work of Venjakob [3]

This allows us to modify the proof of Nichifor–Palvannan to work in this more general setting.

References

[1] Nichifor, A. & Palvannan, B. On free resolutions of Iwasawa modules. *Documenta Mathematica*, 24, pp. 609–662 (2019)
[2] Johnston, H. & Nickel, A. An unconditional proof of the abelian equivariant Iwasawa main conjecture and applications. (arXiv:2010.03186, 2020)
[3] Venjakob, O. A noncommutative Weierstrass preparation theorem and applications to Iwasawa theory. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 559, pp. 153–191 (2003)

Funded by DFG Deutsche Forschungsgemeinschaft German Research Foundation

Background photo: Timo Bobert, Max Greve