Discrete Adic Spaces

Globalisation I

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Recollections and generalisation of the lower shriek functor

Definition

A *pre-analytic ring* $\mathcal{A}$ is a condensed ring $\underline{\mathcal{A}}$ together with a functor $\mathcal{A}[-]$: e.d. sets $\to \mathrm{Mod}^{\mathrm{cond}}_{\underline{\mathcal{A}}}$, taking finite disjoint unions to products, and a natural transformation $\underline{S} \to \mathcal{A}[S]$.

$\underline{A_{\blacksquare}} := A$, $A_{\blacksquare}\left[\varprojlim S_i\right] := \varprojlim A[S_i]$
$\underline{(A,\mathbb{Z})_{\blacksquare}} := A$, $(A,\mathbb{Z})_{\blacksquare}[S] := \mathbb{Z}_{\blacksquare}[S] \otimes_{\mathbb{Z}} A = \mathbb{Z}[S]^{\blacksquare} \otimes_{\mathbb{Z}} A$

Theorem 8.1: $A_{\blacksquare}$ is analytic, $j_! : D(A_{\blacksquare}) \hookrightarrow D((A,\mathbb{Z})_{\blacksquare})$
Theorem 8.2: $f : \mathrm{Spec}\, A \to \mathrm{Spec}\, \mathbb{Z}$, $f_! : D(A_{\blacksquare}) \hookrightarrow D((A,\mathbb{Z})_{\blacksquare}) \hookrightarrow D(\mathbb{Z}_{\blacksquare})$

Let $R \to A$ be a map of finitely generated $\mathbb{Z}$-algebras.
$\underline{(A,R)_{\blacksquare}} := A$, $(A,R)_{\blacksquare}[S] := R_{\blacksquare}[S] \otimes_R A$, note that $(A,A)_{\blacksquare} = A_{\blacksquare}$ by def.

Theorem 8.13: $(A,R)_{\blacksquare}$ is analytic, $j_! : D((A,A)_{\blacksquare} = A_{\blacksquare}) \hookrightarrow D((A,R)_{\blacksquare})$
$f : \mathrm{Spec}\, A \to \mathrm{Spec}\, R$, $f_! : D(A_{\blacksquare}) \hookrightarrow D((A,R)_{\blacksquare}) \hookrightarrow D(R_{\blacksquare})$

Motivation

Goal: globalise the results from talk 9. For this, we will need to be able to glue ⇝ Theorem 9.8 & proof next week

$\underline{(A, R)_\blacksquare} = A$, $(A, R)_\blacksquare[S] = R_\blacksquare[S] \otimes_R A$

Claim

Let $R \to A$ be a map of finitely generated $\mathbb{Z}$-algebras, and A^+ the integral closure of the image of R in A. Then $(A, R)_\blacksquare = (A, A^+)_\blacksquare$.

The underlying rings agree: both are A. Need: $(A, R)_\blacksquare[S] = (A, A^+)_\blacksquare[S]$.
Last week: $\mathbb{Z}_\blacksquare[S] = \prod_I \mathbb{Z}$, $R_\blacksquare[S] = \prod_I R$, $A^+_\blacksquare[S] = \prod_I A^+$.

$$(A, R)_\blacksquare[S] \overset{\text{def}}{=} R_\blacksquare[S] \otimes_R A = \prod_I R \otimes_R A = \prod_I R \otimes_R A^+ \otimes_{A^+} A$$

$$\overset{*}{=} \prod_I A^+ \otimes_{A^+} A = A^+_\blacksquare[S] \otimes_{A^+} A \overset{\text{def}}{=} (A, A^+)_\blacksquare[S]$$

$(*)$ uses that A^+ is finite over R (because R and A are finite over $\mathbb{Z}$) $\square$

Consequence: focus on pairs (A, A^+) where $A^+ \subseteq A$ integrally closed subring ⇝ adic spaces

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Discrete Huber pairs

Definition

Discrete Huber pair: (A, A^+) with A discrete ring, $A^+ \subseteq A$ integrally closed subring.

$(A, A^+)_\blacksquare := \varinjlim (B, B^+)_\blacksquare$ over $(B, B^+) \to (A, A^+)$ maps of discrete Huber pairs with B and B^+ finitely generated $\mathbb{Z}$-algebras.

Definition

A (continuous) *valuation* is $|\cdot| : A \to \Gamma \cup \{0\}$ where Γ is a totally ordered multiplicative abelian group, $\forall \gamma \in \Gamma : 0 < \gamma$, and $|0| = 0$, $|1| = 1$, $|xy| = |x||y|$, $|x + y| \leqslant \max\{|x|, |y|\}$.

It would be more logical to call these (continuous) seminorms, but no.

The adic spectrum

Definition

Support: $\operatorname{supp} |\cdot| := |\cdot|^{-1}(0) = \operatorname{Ker} |\cdot|$

Valuations $|\cdot|, |\cdot|'$ are *equivalent* $\Leftrightarrow$ $|x| \leqslant |y|$ iff $|x|' \leqslant |y|'$ for all $x, y \in A$ $\Leftrightarrow$ supports agree and they are $\leqslant 1$ on the same subset of the residue field $\operatorname{Frac}(A/\operatorname{supp})$. [W, 1.27]

Definition (Adic spectrum as a set)

$\operatorname{Spa}(A, A^+) := \{|\cdot| : A \to \Gamma \cup \{0\} : |A^+| \leqslant 1\}/\text{equivalence}$

Notation: for $x \in \operatorname{Spa}(A, A^+)$ and $f \in A$, write $|f(x)|$ for the valuation of f at x.

Definition (Rational subsets)

$U(\frac{g_1,\ldots,g_n}{f}) := \{x \in \operatorname{Spa}(A, A^+) : \forall i = 1, \ldots, n : |g_i(x)| \leqslant |f(x)| \neq 0\}$

Give $\operatorname{Spa}(A, A^+)$ a topology with the rational subsets as basis.

The rôle of A^+

The smaller A^+, the larger $\mathrm{Spa}(A, A^+)$. The smallest possible A^+ is $\widetilde{\mathbb{Z}} =$ the integral closure of $\mathbb{Z}$ in A. Recall: $U\left(\frac{f}{1}\right) = \{x \in \mathrm{Spa}(A, \widetilde{\mathbb{Z}}) : |f(x)| \leqslant 1\}$

Proposition (Proposition 9.2, adic Nullstellensatz)

There is a bijection

$$\left\{A^+ \subseteq A \textit{ integrally closed}\right\} \leftrightarrow \left\{U = \bigcap_{f \in I} U\left(\frac{f}{1}\right) \subseteq \mathrm{Spa}(A, \widetilde{\mathbb{Z}}) : I \subseteq A\right\}$$

$$A^+ \mapsto \bigcap_{f \in A^+} U\left(\frac{f}{1}\right)$$

$$\{f \in A : \forall x \in U : |f(x)| \leqslant 1\} \leftarrow U$$

The map "$\leftarrow$" is well-defined: the image of U is integrally closed due to non-archimedeanity.

"$\rightarrow$" is surjective: for U given, $A^+ := \langle f \in I \rangle$ has image U.

$$\left\{A^+ \subseteq A \text{ integrally closed}\right\} \leftrightarrow \left\{U = \bigcap_{f \in I} U\left(\tfrac{f}{1}\right) \subseteq \operatorname{Spa}(A, \widetilde{\mathbb{Z}}) : I \subseteq A\right\}$$

$$A^+ \mapsto \bigcap_{f \in A^+} U\left(\tfrac{f}{1}\right)$$

$$\{f \in A : \forall x \in U : |f(x)| \leqslant 1\} \leftarrow U$$

Wts: $A^+ = \{f \in A : \forall x \in \operatorname{Spa}(A, A^+) : |f(x)| \leqslant 1\}$, implies "→" injective.

Clear: $\subseteq$. Let $f \in A \backslash A^+$; wts $f \notin \{f \in A : \forall x \in \operatorname{Spa}(A, A^+) : |f(x)| \leqslant 1\}$.

$f \notin A^+[\frac{1}{f}]$: ↯ otherwise $f = a_0 + \frac{a_1}{f} + \ldots + \frac{a_n}{f^n} \Rightarrow f$ integral over A^+ ↯

$\exists \mathfrak{p} \in \operatorname{Spec}\left(A^+[\frac{1}{f}]\right)$: $\frac{1}{f} \in \mathfrak{p}$. Let $\mathfrak{q} \subseteq \mathfrak{p}$ be a minimal prime.

Let V be a valuation ring of $\operatorname{Frac}(A^+[\frac{1}{f}]/\mathfrak{q})$ dominating $\left(A^+[\frac{1}{f}]/\mathfrak{q}\right)_{\mathfrak{p}/\mathfrak{q}}$.

Recall: for $R \subseteq S$ local rings, R *dominates* S if $\mathfrak{m}_S \cap R = \mathfrak{m}_R$. Local subrings of a field are inductively ordered wrt domination, valuation rings are the maximal elements.

Valuation on $V \rightsquigarrow$ valuation s on $A^+[\frac{1}{f}]$. $\operatorname{supp} s = \mathfrak{q}$, $s(A^+) \leqslant 1$ (V valuation ring), $s(\mathfrak{p}) < 1$ (domination). Thus $\frac{1}{f} \in \mathfrak{p} \Rightarrow s(f) > 1$.

As $A^+[\frac{1}{f}] \subseteq A_f$, there is a prime over $\mathfrak{q}$, hence there is a valuation t on A_f extending s. Now $t|_A \in \operatorname{Spa}(A, A^+)$ and $t|_A(f) > 1$. $\square$

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Proposition (Proposition 9.3)

Let $\varphi : (A, A^+) \to (B, B^+)$ *be a map of discrete Huber spaces such that the induced map factors as*

$$\begin{tikzcd}
\mathrm{Spa}(B, B^+) \arrow[r, "\mathrm{Spa}(\varphi)"] \arrow[dr] & \mathrm{Spa}(A, A^+) \\
& U = U\left(\frac{g_1, \dots, g_n}{f}\right) \arrow[u, hook]
\end{tikzcd}$$

Then φ *factors uniquely as*

$$\begin{tikzcd}
(A, A^+) \arrow[r, "\varphi"] \arrow[d, hook] & (B, B^+) \\
\left(A\left[\frac{1}{f}\right], \widetilde{A^+\left[\frac{g_1}{f}, \dots, \frac{g_n}{f}\right]}\right) \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

Moreover, $\mathrm{Spa}\left(A\left[\frac{1}{f}\right], \widetilde{A^+\left[\frac{g_1}{f}, \dots, \frac{g_n}{f}\right]}\right) \xrightarrow{\cong} U \hookrightarrow \mathrm{Spa}(A, A^+)$*.*

Here $\widetilde{A^+\left[\frac{g_1}{f}, \dots, \frac{g_n}{f}\right]}$ is the integral closure of $A^+\left[\frac{g_1}{f}, \dots, \frac{g_n}{f}\right]$ in $A\left[\frac{1}{f}\right]$

Upshot: $U\left(\frac{g_1, \dots, g_n}{f}\right)$ determines $A[\frac{1}{f}]$ and $\widetilde{A^+[\frac{g_1}{f}, \dots, \frac{g_n}{f}]}$.

$\varphi : (A, A^+) \to (B, B^+)$, $\mathrm{Spa}(\varphi) : \mathrm{Spa}(B, B^+) \to U(\frac{g}{f}) \hookrightarrow \mathrm{Spa}(A, A^+)$

Claim: $\varphi(f) \in B^\times$.

Every prime of B gives rise to a valuation:

$$\begin{aligned} \psi : \mathrm{Spec}\, B &\hookrightarrow \mathrm{Spa}(B, B^+) \\ \mathfrak{p} &\mapsto (B \twoheadrightarrow B/\mathfrak{p} \hookrightarrow \mathrm{Frac}(B/\mathfrak{p}) \to \{0, 1\}) \end{aligned}$$

Proof of Claim: ↯ $\varphi(f) \in B - B^\times$. Let $(0) \neq \overline{\mathfrak{p}} \in \mathrm{Spec}(B/\varphi(f)B) \neq \{(0)\}$.
This $\overline{\mathfrak{p}}$ lifts to $\mathfrak{p} \in \mathrm{Spec}(B)$. Then $|f(\psi(\mathfrak{p}))| = 0$.
But since $\mathrm{Spa}(\varphi)$ factors over U, for all $y \in \mathrm{Spa}(B, B^+)$ we have
$|\varphi(f)(y)| = |f(\mathrm{Spa}(\varphi)(y))| > 0$ ↯ Hence the Claim.
Claim $\Rightarrow$ φ uniquely extends to $A[\frac{1}{f}] \to B$.
$\mathrm{Spa}(\varphi)$ factors over $U \Rightarrow |\varphi(\frac{g_i}{f})(y)| = |\left(\frac{g_i}{f}\right)(\mathrm{Spa}(\varphi)(y))| \leqslant 1$
Proposition 9.2 $\Rightarrow \varphi(\frac{g_i}{f}) \in B^+ \Rightarrow A^+[\frac{g}{f}] \to B^+$
Thus φ induces a unique $\left(A\left[\frac{1}{f}\right], \overline{A^+\left[\frac{g_1}{f}, \dots, \frac{g_n}{f}\right]}\right) \to (B, B^+)$. $\square$

By Proposition 9.3, it makes sense to make the following definition:

Definition

For $U = U(\frac{g}{f}) \subseteq X = \mathrm{Spa}(A, A^+)$, let $\mathcal{O}_X(U) := A[\frac{1}{f}]$, $\mathcal{O}^+(U) := \widetilde{A^+[\frac{g}{f}]}$

Proposition (Proposition 9.4)

1. $\forall x \in X$, $f \mapsto |f(x)|$ *uniquely extends to* $\mathcal{O}_{X,x}$.
2. $\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : \forall x \in U, |f(x)| \leqslant 1\}$
3. $\mathcal{O}_X$ *and* $\mathcal{O}_X^+$ *are sheaves.*

Assertion 1. Let $U = U(\frac{g}{f}) \subseteq X$.

Proposition 9.3 $\Rightarrow U \cong \mathrm{Spa}(A[\frac{1}{f}], \widetilde{A^+[\frac{g}{f}]})$.

Hence every $x \in U$ gives rise to a valuation on $A[\frac{1}{f}] = \mathcal{O}_X(U)$.

Valuations extend uniquely to the direct limit $\mathcal{O}_{X,x} = \varinjlim \mathcal{O}_X(U)$. □

Assertion 2: $\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : \forall x \in U, |f(x)| \leqslant 1\}$

Let $U = U(\frac{g}{f}) \subseteq X$.

$$\mathcal{O}_X^+(U) \underset{\text{def}}{=} \widetilde{A^+\left[\frac{g}{f}\right]} \underset{9.2}{=} \left\{h \in A\left[\frac{1}{f}\right] : \forall x \in \operatorname{Spa}\left(A\left[\frac{1}{f}\right], \widetilde{A^+\left[\frac{g}{f}\right]}\right), |h(x)| \leqslant 1\right\}$$
$$\underset{9.3}{=} \left\{h \in A\left[\frac{1}{f}\right] : \forall x \in U, |h(x)| \leqslant 1\right\} \qquad \square$$

Assertion 3: $\mathcal{O}_X$ and $\mathcal{O}_X^+$ are sheaves.

Both are presheaves.
Assertion 2 shows that $\mathcal{O}_X^+$ is a sheaf whenever $\mathcal{O}_X$ is (uniqueness comes from $\mathcal{O}_X$, and gluing sections $|\cdot| \leqslant 1$ will yield in a section $|\cdot| \leqslant 1$.)
Hence we may focus on $\mathcal{O}_X$.

$$\begin{aligned}\psi : \mathsf{Spec}(A) &\hookrightarrow \mathsf{Spa}(A, A^+)\\ \mathfrak{p} &\mapsto (A \twoheadrightarrow A/\mathfrak{p} \hookrightarrow \mathsf{Frac}(A/\mathfrak{p}) \to \{0, 1\})\end{aligned}$$

For $U = U(\frac{g}{f}) \subseteq X = \mathsf{Spa}(A, A^+)$: $\psi^{-1}(U) = D(f) = \{\mathfrak{p} \in \mathsf{Spec}\, A : f \notin \mathfrak{p}\}$

Indeed: if $\mathfrak{p} \in D(f)$ then $|f(\psi(\mathfrak{p}))| = 1 \geqslant |g_i(\psi(\mathfrak{p}))|$, so $\psi(\mathfrak{p}) \in U$.

Conversely: if $\mathfrak{p} \notin D(f)$ then $|f(\psi(\mathfrak{p}))| = 0$, hence $\psi(\mathfrak{p}) \notin U$.

$\psi_* \mathcal{O}_{\mathsf{Spec}\, A} = \mathcal{O}_{\mathsf{Spa}(A, A^+)}$ as presheaves:

$$\left(\psi_* \mathcal{O}_{\mathsf{Spec}\, A}\right)(U) = \mathcal{O}_{\mathsf{Spec}\, A}\left(\psi^{-1} U\right) = \mathcal{O}_{\mathsf{Spec}\, A}(D(f)) = A\left[\frac{1}{f}\right] = \mathcal{O}_X(U)$$

$\mathcal{O}_{\mathsf{Spec}\, A}$ is a sheaf, and pushforward preserves sheaves $\Rightarrow$ $\mathcal{O}_X$ is a sheaf. $\square$

Definition (Discrete adic space)

A *discrete adic space* is a triple $(X, \mathcal{O}_X, |\cdot(x)|_{x\in X})$ where

- X is a topological space,
- $\mathcal{O}_X$ is a sheaf of rings,
- $\forall x \in X$, $|\cdot(x)|$ is an equivalence class of valuations on $\mathcal{O}_{X,x}$,
- locally looks like $(\mathrm{Spa}(A, A^+), \mathcal{O}_{\mathrm{Spa}(A,A^+)}, |\cdot(x)|_{x\in \mathrm{Spa}(A,A^+)})$ for (A, A^+) discrete Huber pair.

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The functors $(-)^{\mathrm{ad}}$ and $(-)^{\mathrm{ad}/R}$

Fully faithful functors $(-)^{\mathrm{ad}}, (-)^{\mathrm{ad}/R} : \mathsf{Schemes} \to \mathsf{DiscreteAdicSpaces}$

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, mapsto] \arrow[d, mapsto] & \operatorname{Spa}(A, A) \arrow[dl, hook'] & {} \arrow[r, rightsquigarrow, "\text{glue}"] & {} & X \arrow[r, mapsto] \arrow[d, mapsto] & X^{\mathrm{ad}} \arrow[dl] \\
\operatorname{Spa}(A, \widetilde{R}) & & & & X^{\mathrm{ad}/R} &
\end{tikzcd}$$

As before: $R \to A$ a map of finitely generated $\mathbb{Z}$-algebras.

Claim (Description of points in terms of scheme-theoretic data)

$$X^{\mathrm{ad}} = \frac{\{\operatorname{Spec} V \to X : V \textit{ valuation ring}\}}{\{(\operatorname{Spec} V \to X) \sim (\operatorname{Spec} W \to \operatorname{Spec} V \to X) : V \to W \textit{ faithfully flat}\}}$$

$$X^{\mathrm{ad}/R} = \left\{ \begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} V \arrow[r] & \operatorname{Spec} R
\end{tikzcd} : K = \operatorname{Frac}(V), V \textit{ valuation ring} \right\} \Big/ \sim$$

For the proof, we will need a lemma. (Credits to Alex Mathers on MSE.)

Lemma

A map $V \to W$ between valuation rings is faithfully flat iff it is injective and local.

Proof of Lemma: combine Lemma 0539 and Lemma 00HP of [Stacks]. □

Proof of Prop.: only prove the first assertion. Suffices to check locally:

$$\begin{aligned} \operatorname{Spa}(A,A) &\overset{?}{=} \{\operatorname{Spec} V \to \operatorname{Spec} A\}/\{\operatorname{Spec} W \to \operatorname{Spec} V : V \to W \text{ faithf. flat}\} \\ &= \{A \to V : V \text{ valuation ring}\}/\{V \to W \text{ faithfully flat}\} \\ |\cdot|_V \circ \varphi &\mapsfrom (\varphi : A \to V \text{ with valuation } |\cdot|_V \text{ on } V) \\ v &\mapsto (A \twoheadrightarrow A/\operatorname{supp} v \hookrightarrow V_v) \end{aligned}$$

where V_v is the valuation ring of the induced valuation on $\operatorname{Frac}(A/\operatorname{supp} v)$.

$$\begin{aligned} \mathrm{Spa}(A,A) &\overset{?}{=} \{A \to V : V \text{ valuation ring}\}/\{V \to W \text{ faithfully flat}\} \\ |\cdot|_V \circ \varphi &\leftarrow\!\shortmid (\varphi : A \to V \text{ with valuation } |\cdot|_V \text{ on } V) \\ v &\mapsto (A \twoheadrightarrow A/\operatorname{supp} v \hookrightarrow V_v) \end{aligned}$$

where V_v is the valuation ring of the induced valuation on $\mathrm{Frac}(A/\operatorname{supp} v)$.
Lemma: $V \to W$ is faithfully flat iff injective and local.

First we check that these maps are **well-defined**.

"$\mapsto$": the image depends only on $\operatorname{supp} v$, and equivalent valuations have the same support. [W, 1.27]

"$\leftarrow\!\shortmid$": let $\iota : V \to W$ be faithfully flat. Want: $|\cdot|_V \circ \varphi = |\cdot|_V \circ \iota \circ \varphi$.
Claim: $\mathrm{Frac}(V) \cap W = V$. Indeed, the Lemma implies $\mathrm{Frac}(V) \cap W \supseteq V$ and $\mathfrak{m}_{\mathrm{Frac}(V) \cap W} = \mathfrak{m}_V$; both are valuation rings, hence maximal wrt domation $\Rightarrow$ equality.
$\operatorname{supp}(|\cdot|_V \circ \varphi) = \operatorname{supp}(|\cdot|_V \circ \iota \circ \varphi)$ because ι is injective (Lemma).
Valuations are equivalent $\Leftrightarrow$ supports agree and they are $\leqslant 1$ on the same subset of the residue field $\mathrm{Frac}(A/\operatorname{supp})$. [W, 1.27]

$$\begin{aligned} \operatorname{Spa}(A,A) &\overset{?}{=} \{A \to V : V \text{ valuation ring}\}/\{V \to W \text{ faithfully flat}\} \\ |\cdot|_V \circ \varphi &\leftarrow\!\shortmid (\varphi : A \to V \text{ with valuation } |\cdot|_V \text{ on } V) \\ v &\mapsto (A \twoheadrightarrow A/\operatorname{supp} v \hookrightarrow V_v) \end{aligned}$$

where V_v is the valuation ring of the induced valuation on $\operatorname{Frac}(A/\operatorname{supp} v)$.
Lemma: $V \to W$ is faithfully flat iff injective and local.

Now we check that the two maps are **inverse to one another**.

It is clear that $v \mapsto (A \twoheadrightarrow A/\operatorname{supp} v \hookrightarrow V_v) \mapsto v$.

Converse: $(\varphi : A \to V) \mapsto (v := |\cdot|_V \circ \varphi) \mapsto (\varphi_v : A \twoheadrightarrow A/\operatorname{supp} v \hookrightarrow V_v)$
Want to show: φ and φ_v agree up to a faithfully flat map.
φ induces $A/\operatorname{supp} v \hookrightarrow V$ because $\operatorname{Ker} \varphi \subseteq \operatorname{Ker}(|\cdot|_V \circ \varphi) = \operatorname{supp} v$
Pass to fraction fields: $\operatorname{Frac}(A/\operatorname{supp} v) \hookrightarrow \operatorname{Frac}(V)$
We get $V_v \hookrightarrow V$ since $V_v \subseteq \operatorname{Frac}(A/\operatorname{supp} v)$ is a valuation ring
$\mathfrak{m}_{V_v} \hookrightarrow \mathfrak{m}_V$ by construction
Apply the Lemma: $V_v \to V$ is faithfully flat. □

As before: $R \to A$ a map of finitely generated $\mathbb{Z}$-algebras.

Proposition (Proposition 9.6)

$X^{\mathrm{ad}} \to X^{\mathrm{ad}/R}$ is

- *an open immersion if X is separated and of finite type over* $\operatorname{Spec} R$*;*
- *an isomorphism if X is proper over* $\operatorname{Spec} R$*.*

Claim: $X^{\mathrm{ad}} \to X^{\mathrm{ad}/R}$ is an open immersion locally on X^{ad}.

Indeed, it is sufficient to show this for $X = \operatorname{Spec} A$, i.e. checking that $\varphi : \operatorname{Spa}(A, A) \to \operatorname{Spa}(A, \widetilde{R})$ is open immersion.

- φ is injective
- $\varphi(\operatorname{Spa}(A, A)) = \bigcup_{g \in A} U_{\operatorname{Spa}(A, \widetilde{R})}(\frac{g}{1})$ open
- $\varphi^{-1}\mathcal{O}^{(+)}_{\operatorname{Spa}(A, \widetilde{R})}(U) = \mathcal{O}^{(+)}_{\operatorname{Spa}(A, A)}(U)$ follows from definition of $\mathcal{O}$, $\mathcal{O}^{+}$

The Claim is proven.

Proposition (Proposition 9.6)

$X^{\mathrm{ad}} \to X^{\mathrm{ad}/R}$ is

- *an open immersion if X is separated and of finite type over $\operatorname{Spec} R$;*
- *an isomorphism if X is proper over $\operatorname{Spec} R$.*

$X^{\mathrm{ad}} \to X^{\mathrm{ad}/R}$ is an open immersion locally on X^{ad}. Hence it is open and the induced map on stalks is an iso.

Recall the description of points:

$$X^{\mathrm{ad}} = \frac{\{\operatorname{Spec} V \to X : V \text{ valuation ring}\}}{\{(\operatorname{Spec} V \to X) \sim (\operatorname{Spec} W \to \operatorname{Spec} V \to X) : V \to W \text{ faithfully flat}\}}$$

$$X^{\mathrm{ad}/R} = \left\{ \begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} V \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} R \end{tikzcd} : K = \operatorname{Frac}(V),\ V \text{ valuation ring} \right\} \Big/ \sim$$

Valuative criterion of separatedness $\Rightarrow$ $X^{\mathrm{ad}} \to X^{\mathrm{ad}/R}$ is injective on points if X is separated oft. Hence it is an open immersion.

Valuative criterion of properness $\Rightarrow$ $X^{\mathrm{ad}} \to X^{\mathrm{ad}/R}$ is bijective on points if X is proper. Hence it is an isomorphism. $\square$

How will we use Proposition 9.6?

Proposition (Proposition 9.6)

$X^{\mathrm{ad}} \to X^{\mathrm{ad}/R}$ is

- *an open immersion if X is separated and of finite type over $\operatorname{Spec} R$;*
- *an isomorphism if X is proper over $\operatorname{Spec} R$.*

Theorem 8.13(i): $(A, R)_{\blacksquare}$ is analytic. Today: $(A, R)_{\blacksquare} = (A, A^+)_{\blacksquare}$ where $R \to A$ map of finitely generated $\mathbb{Z}$-algebras and A^+ is the integral closure.

(ii-iii): $f : \operatorname{Spec} A \to \operatorname{Spec} R$ map of affine finite type $\mathbb{Z}$-schemes,
$j_! : D((A, A)_{\blacksquare} = A_{\blacksquare}) \hookrightarrow D((A, R)_{\blacksquare})$, $f_! : D(A_{\blacksquare}) \hookrightarrow D((A, R)_{\blacksquare}) \hookrightarrow D(R_{\blacksquare})$

To prove global versions of the theorems in Viktor's talk, we will do gluing (Ulrich's talk). The cat of modules on X^{ad} will be glued together from $D(A_{\blacksquare})$'s, those on $X^{\mathrm{ad}/R}$ from $D((A, R)_{\blacksquare})$'s.

Proposition 9.6 $\Rightarrow$ $j_!$ is the identity when f is proper.

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The obstruction to gluing $(A, A^+)_\blacksquare$-modules

Claim

For $U \subseteq X = \mathrm{Spa}(A, A^+)$ rational, $- \otimes_{(A,A^+)_\blacksquare} (\mathcal{O}_X(U), \mathcal{O}_X^+(U))_\blacksquare$ is not exact.

Example (Counterexample)

$(A, A^+) := (\mathbb{F}_p[T], \mathbb{F}_p)$, $U := U(\frac{T}{1}) = \{|T| \leqslant 1\}$,
$(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) = (\mathbb{F}_p[T][\frac{1}{1}], \widetilde{\mathbb{F}_p[\frac{T}{1}]}) = (\mathbb{F}_p[T], \mathbb{F}_p[T])$
Then $\mathbb{F}_p[T] \hookrightarrow \mathbb{F}_p((T^{-1}))$ does not remain injective under base change.

Credits for the proof go to Chirantan and Robin.

Remark 1 (Proposition 7.7, Gürkan's talk): let $f : \mathcal{A} \to \mathcal{B}$ be a map of analytic rings. This gives rise to $\mathrm{Mod}_{\mathcal{A}}^{\mathrm{cond}} \to \mathrm{Mod}_{\mathcal{B}}^{\mathrm{cond}}$, $M \mapsto M \otimes_{\mathcal{A}} \mathcal{B}$.
In particular, if $M = \mathcal{A}[S]$ then $M \otimes_{\mathcal{A}} \mathcal{B} = \mathcal{B}[S]$.

Remark 1: if $M = \mathcal{A}[S]$ then $M \otimes_{\mathcal{A}} \mathcal{B} = \mathcal{B}[S]$.

Remark 2: we are working with the following analytic rings:

$$\begin{array}{lll}
\mathcal{A} = (\mathbb{F}_p[T], \mathbb{F}_p)_\blacksquare & \underline{\mathcal{A}} = \underline{\mathbb{F}_p[T]} & \mathcal{A}[S] = \varprojlim_i \mathbb{F}_p[S_i] \otimes_{\underline{\mathbb{F}_p}} \underline{\mathbb{F}_p[T]} \\
\mathcal{B} = (\mathbb{F}_p[T], \mathbb{F}_p[T])_\blacksquare & \underline{\mathcal{B}} = \underline{\mathbb{F}_p[T]} & \mathcal{B}[S] = \varprojlim_i \underline{\mathbb{F}_p[T]}[S_i]
\end{array}$$

The claim: in $\mathrm{Mod}^{\mathrm{cond}}_{(\mathbb{F}_p[T],\mathbb{F}_p[T])_\blacksquare}$, we have:

$$\begin{tikzcd}
\underline{\mathbb{F}_p[T]} \underset{(\mathbb{F}_p[T],\mathbb{F}_p)_\blacksquare}{\otimes} (\mathbb{F}_p[T], \mathbb{F}_p[T])_\blacksquare \arrow[r] \arrow[d, "\cong"] & \underline{\mathbb{F}_p((T^{-1}))} \underset{(\mathbb{F}_p[T],\mathbb{F}_p)_\blacksquare}{\otimes} (\mathbb{F}_p[T], \mathbb{F}_p[T])_\blacksquare \arrow[d, equal] \\
\underline{\mathbb{F}_p[T]} \arrow[r] & 0
\end{tikzcd}$$

The left isomorphism:

$\mathcal{A}[*] = (\mathbb{F}_p[T], \mathbb{F}_p)_\blacksquare[*] = \varprojlim \underline{\mathbb{F}_p}[*] \otimes_{\underline{\mathbb{F}_p}} \underline{\mathbb{F}_p[T]} = \underline{\mathbb{F}_p} \otimes_{\underline{\mathbb{F}_p}} \underline{\mathbb{F}_p[T]} = \underline{\mathbb{F}_p[T]}$

Apply Remark 1 with $S := *$ and $M := \underline{\mathbb{F}_p[T]}$:

$\underline{\mathbb{F}_p[T]} \otimes_{\mathcal{A}} \mathcal{B} = (\mathbb{F}_p[T], \mathbb{F}_p[T])_\blacksquare[*] = \varprojlim \underline{\mathbb{F}_p[T]}[*] = \underline{\mathbb{F}_p[T]}$

The right isomorphism: Observation 8.6 (Viktor's talk, replace $\mathbb{Z}$ by $\mathbb{F}_p$):

$$0 \to \mathbb{F}_p[[U]][T] \xrightarrow{UT-1} \mathbb{F}_p[[U]][T] \to \mathbb{F}_p((T^{-1})) \to 0 \qquad (*)$$

$\mathbb{F}_p((T^{-1}))$ becomes 0 after base change $\Leftrightarrow$ $UT-1$ becomes an iso.
$S := \mathbb{N} \cup \infty$ one-point compactification.

$$\begin{aligned}(\mathbb{F}_p[T], \mathbb{F}_p)_{\blacksquare}[S]/(0=\infty) &\overset{\text{def}}{=} \varprojlim_i \mathbb{F}_p[S_i] \otimes_{\mathbb{F}_p} \mathbb{F}_p[T]/(0=\infty) \\ &\cong \left(\varprojlim_i \mathbb{F}_p[S_i]/(0=\infty) \right) \otimes_{\mathbb{F}_p} \mathbb{F}_p[T] \end{aligned}$$

[Sch2, II, p.12]

$$\cong \mathbb{F}_p[[U]] \otimes_{\mathbb{F}_p} \mathbb{F}_p[T] \cong \mathbb{F}_p[[U]][T]$$

Remark 1: $(\mathbb{F}_p[T], \mathbb{F}_p)_{\blacksquare}[S]$ gets base changed to $(\mathbb{F}_p[T], \mathbb{F}_p[T])_{\blacksquare}[S]$
Hence $\mathbb{F}_p[[U]][T]$ becomes

$$(\mathbb{F}_p[T], \mathbb{F}_p[T])_{\blacksquare}[S]/(0=\infty) \overset{\text{def}}{=} \varprojlim \mathbb{F}_p[T][S_i]/(0=\infty) = \mathbb{F}_p[T][[U]]$$

$UT-1$ is a unit in $\mathbb{F}_p[T][[U]]$. $\square$

Claim (again)

For $U \subseteq X = \mathrm{Spa}(A, A^+)$ *rational,* $- \otimes_{(A,A^+)_\blacksquare} (\mathcal{O}_X(U), \mathcal{O}_X^+(U))_\blacksquare$ *is not exact.*

This shows that in fact we should work with derived tensor products, and glue on the level of derived categories. But that will necessitate using ∞-categories $\rightsquigarrow$ next talk (Jochen).

Table of Contents

Description of $D((\mathcal{O}_X(U), \mathcal{O}_X^+(U))_\blacksquare)$ without ∞-categories

X separated discrete adic space, $X = \bigcup_{i=1}^n U_i$, $U_i = \mathrm{Spa}(A_i, A_i^+)$ cover

For $I \subseteq \{1, \dots, n\}$, let $U_I := \bigcap_{i \in I} U_i = \mathrm{Spa}(A_I, A_I^+)$ for some (A_I, A_I^+)

We define the following category:

$$\begin{aligned}\mathrm{Cond}_{\{U_I\}}(\mathcal{O}_X) = \{&M_- : I \mapsto M_I \text{ condensed } A_I\text{-module} \\ &\text{with } M_I \otimes_{A_I} A_J \to M_J \text{ for all } I \subseteq J\} \\ = \{&\text{presheaves of condensed } \mathcal{O}_X\text{-modules on } \{U_I : I\}\}\end{aligned}$$

Scholze: this is an abelian category with compact projective generators.

We have $D((\mathcal{O}_X(U), \mathcal{O}_X^+(U))_\blacksquare) \to D(\mathrm{Cond}_{\{U_I\}}(\mathcal{O}_X))$ fully faithful, and the essential image is:

$$\Big\{ M \in D(\mathrm{Cond}_{\{U_I\}}(\mathcal{O}_X)) : \forall I, M_I \in D((A_I, A_I^+)_\blacksquare) \text{ and } \\ M_I \otimes^L_{(A_I, A_I^+)_\blacksquare} (A_J, A_J^+)_\blacksquare \to M_J \text{ equivalence for all } I \subseteq J \Big\}$$

Independence of $\{U_I\}$ will follow from Theorem 9.8 in the next talk.

Thank you for your attention!

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