Some reduction steps and the Ritter–Weiss module

Talk 5

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1 December 2020

Where were we?

H/F abelian extension of number fields, H CM, F totally real,
$\mathrm{Gal}(H/F) = G \simeq G_p \times G'$

$\Sigma := \{v \in S_{\mathrm{ram}} : v \mid p\} \cup S_\infty$
$\Sigma' := \{v \in S_{\mathrm{ram}} : v \nmid p\} \cup T$

$$\prod_{w \notin \Sigma_H \cup \Sigma'_H} \mathbb{Z} \to \mathrm{Hom}_{\mathbb{Z}}\left(H^\times_{\Sigma'}, \mathbb{Z}\right) \qquad \mathrm{Sel}_\Sigma^{\Sigma'}(H) := \text{cokernel}$$
$$(x_w) \mapsto \sum_w x_w \,\mathrm{ord}_w(-)$$

$\mathcal{O}$ finite extension of $\mathbb{Z}_p$
$\Psi \subseteq \widehat{G}$ subset of characters

$$\mathcal{O}[G] \to \prod_{\psi \in \Psi} \mathcal{O}_\psi \qquad R_\Psi := \text{image} \qquad \text{'character group ring'}$$
$$x \mapsto (\psi(x))_{\psi \in \Psi}$$

$\chi \in \widehat{G'}$ $\quad \Psi_\chi := \{\psi \in \widehat{G} : \psi|_{G'} = \chi\}$ 'characters belonging to χ' $\quad R := R_\chi := R_{\Psi_\chi}$

Theorem 3.3 (Goal)

The $\mathbb{Z}_p[G]^-$-module $\mathrm{Sel}_\Sigma^{\Sigma'}(H)_p^-$ is quadratically presented, and

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}\left(\mathrm{Sel}_\Sigma^{\Sigma'}(H)_p^-\right) = \Theta^\#_{\Sigma,\Sigma'}\mathbb{Z}_p[G]^-$$

Part I

Reduction steps

Removing primes above p from Σ'

$\Sigma := \{v \in S_{\mathrm{ram}} : v \mid p\} \cup S_\infty$
$\Sigma' := \{v \in S_{\mathrm{ram}} : v \nmid p\} \cup T$
$\Sigma'' := \Sigma' - \{v \in T : v \mid p\}$

Lemma 4.1 (Removing primes above p from Σ')

$$\mathrm{Sel}_\Sigma^{\Sigma'}(H)_R \simeq \mathrm{Sel}_\Sigma^{\Sigma''}(H)_R \text{ and } \Theta^{\#}_{\Sigma,\Sigma'}R = \Theta^{\#}_{\Sigma,\Sigma''}R$$

$$0 \to \mathrm{Sel}_\Sigma^{\Sigma''}(H)_R \to \mathrm{Sel}_\Sigma^{\Sigma'}(H)_R \to \left(\prod_{v\in\Sigma'-\Sigma''}\prod_{w\mid v}(\mathcal{O}_H/w)^\times\right)^\vee_R \to 0$$

(Proof on the next slide.)

Last term: $w \mid v \mid p \Rightarrow p \nmid \#(\mathcal{O}_H/w)^\times$

$\Rightarrow$ killed by $(- \otimes R)$ because R is a $\mathbb{Z}_p$-algebra

$\Rightarrow \mathrm{Sel}_\Sigma^{\Sigma'}(H)_R \simeq \mathrm{Sel}_\Sigma^{\Sigma''}(H)_R$

$\Sigma := \{v \in S_{\mathrm{ram}} : v \mid p\} \cup S_\infty$ $\qquad$ $\Sigma'' := \Sigma' - \{v \in T : v \mid p\}$

$\Sigma' := \{v \in S_{\mathrm{ram}} : v \nmid p\} \cup T$

The exact sequence from the previous slide

$$0 \to \mathrm{Sel}_\Sigma^{\Sigma''}(H)_R \to \mathrm{Sel}_\Sigma^{\Sigma'}(H)_R \to \left(\prod_{v \in \Sigma' - \Sigma''} \prod_{w \mid v} (\mathcal{O}_H/w)^\times \right)_R^\vee \to 0$$

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^-_{H,\Sigma} & \longrightarrow & \mathrm{Sel}_\Sigma^{\Sigma''}(H)^- & \longrightarrow & \mathrm{Cl}^{\Sigma''}(H)^{\vee,-} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y^-_{H,\Sigma} & \longrightarrow & \mathrm{Sel}_\Sigma^{\Sigma'}(H)^- & \longrightarrow & \mathrm{Cl}^{\Sigma'}(H)^{\vee,-} & \longrightarrow & 0 \\
 & & \downarrow & & \twoheaddownarrow & & \twoheaddownarrow & & \\
 & \longrightarrow & 0 & \longrightarrow & \left(\prod (\mathcal{O}_H/w)^\times\right)^{\vee,-} & \longrightarrow & \left(\prod (\mathcal{O}_H/w)^\times\right)^{\vee,-} & \longrightarrow & 0
\end{array}$$

$$\Theta^{\#}_{\Sigma,\Sigma'} = \Theta^{\#}_{\Sigma,\Sigma''} \prod_{v \in \Sigma' - \Sigma''} (1 - \sigma_v \mathrm{N}v)$$

Proof: apply $\psi \in \widehat{G}$ and use defining property of Θ.

$$\begin{aligned}\psi\left(\Theta^{\#}_{\Sigma,\Sigma'}\right) = L_{\Sigma,\Sigma'}(0,\psi) &= L_{\Sigma}(0,\psi) \prod_{v \in \Sigma'} \left(1 - \psi(v)\mathrm{N}v^{1-0}\right) \\ &= \underbrace{L_{\Sigma}(0,\psi) \prod_{v \in \Sigma''} (1 - \psi(v)\mathrm{N}v)}_{L_{\Sigma,\Sigma''}(0,\psi) = \psi\left(\Theta^{\#}_{\Sigma,\Sigma''}\right)} \prod_{v \in \Sigma' - \Sigma''} \underbrace{(1 - \psi(v)\mathrm{N}v)}_{\psi(1 - \sigma_v \mathrm{N}v)}\end{aligned}$$

$1 - \sigma_v \mathrm{N}v$ has unit image in R:

$$(1 - \sigma_v \mathrm{N}v) \sum_{i=0}^{\mathrm{ord}(\sigma_v)-1} \sigma_v^i (\mathrm{N}v)^i = 1 - \underbrace{\sigma_v^{\mathrm{ord}(\sigma_v)}}_{1} \cdot \underbrace{(\mathrm{N}v)^{\mathrm{ord}(\sigma_v)}}_{\text{in max ideal of } \mathcal{O}_\chi}$$

$\Rightarrow \Theta^{\#}_{\Sigma,\Sigma'} R = \Theta^{\#}_{\Sigma,\Sigma''} R$ □

Lemma (Burns–Kurihara–Sano)

Let $S' \supseteq \Sigma$ disjoint from Σ' such that $\mathrm{Cl}_{S'}^{\Sigma'}(H) = 1$. Then we have

$$0 \to \prod_{w \in S'_H - \Sigma_H} \mathbb{Z} \to \mathrm{Hom}_{\mathbb{Z}}\left(\mathcal{O}_{H,S',\Sigma'}^{\times}, \mathbb{Z}\right) \to \mathrm{Sel}_{\Sigma}^{\Sigma'}(H) \to 0$$

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & 0 & \longrightarrow & {\color{red}0} & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{w \notin S'_H \cup \Sigma'_H} \mathbb{Z} & \longrightarrow & \prod_{w \notin \Sigma_H \cup \Sigma'_H} \mathbb{Z} & \longrightarrow & \prod_{w \in S'_H \cup \Sigma_H} \mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{w \notin S'_H \cup \Sigma'_H} \mathbb{Z} & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}\left(H_{\Sigma'}^{\times}, \mathbb{Z}\right) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}}\left(\mathcal{O}_{H,S',\Sigma'}^{\times}, \mathbb{Z}\right) & \longrightarrow & 0 \\
 & & \downarrow & & \twoheadrightarrow & & \twoheadrightarrow & & \\
 & \longrightarrow & 0 & \longrightarrow & \mathrm{Sel}_{\Sigma}^{\Sigma'}(H) & \longrightarrow & {\color{red}\mathrm{Sel}_{\Sigma}^{\Sigma'}(H)} & \longrightarrow & 0
\end{array}$$

Third row: $\mathbb{Z}$-dual of $0 \to \mathcal{O}_{H,S',\Sigma'}^{\times} \to H_{\Sigma'}^{\times} \xrightarrow{\oplus \mathrm{ord}_w} \bigoplus_{w \in S'_H \cup \Sigma'_H} \mathbb{Z} \to \mathrm{Cl}_{S'}^{\Sigma'}(H) \to 0$

Vertical arrows: $(x_w)_w \mapsto (- \mapsto \sum_w \mathrm{ord}_w(-) x_w)$ □

Making χ faithful, step 1

Recall: $G = \mathrm{Gal}(H/F) = G' \times G_p$ $\quad \chi \in \widehat{G'}$, $H_\chi :=$ fixed field of $\mathrm{Ker}\,\chi \subseteq G' \subseteq G$

Lemma 4.2 (Replacing H by H_χ)

$\mathrm{Sel}_\Sigma^{\Sigma'}(H)_R \cong \mathrm{Sel}_\Sigma^{\Sigma'}(H_\chi)_R$ canonical isomorphism.

Previous lemma: $0 \to \prod_{w \in S'_H - \Sigma_H} \mathbb{Z} \to \mathrm{Hom}_{\mathbb{Z}}\left(\mathcal{O}^\times_{H,S',\Sigma'}, \mathbb{Z}\right) \to \mathrm{Sel}_\Sigma^{\Sigma'}(H) \to 0$

$$
\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & 0 & \longrightarrow & \\
 & & & & \downarrow & & \downarrow & & \\
 & & Y_{H,S'-\Sigma} \otimes R & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}\left(\left(\mathcal{O}^\times_{H,S',\Sigma'} \otimes \mathcal{O}\right)^{G'=\chi}, \mathcal{O}\right) & \longrightarrow & \mathrm{Sel}_\Sigma^{\Sigma'}(H)_R & \longrightarrow & 0 \\
 & & \twoheaddownarrow & & \downarrow\cong & & \downarrow & & \\
0 & \longrightarrow & Y_{H_\chi,S'-\Sigma} \otimes R & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}\left(\left(\mathcal{O}^\times_{H_\chi,S',\Sigma'} \otimes \mathcal{O}\right)^{G'=\chi}, \mathcal{O}\right) & \longrightarrow & \mathrm{Sel}_\Sigma^{\Sigma'}(H_\chi)_R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

□

Making χ faithful, step 2

$$\Sigma(\chi) := \{v \mid p : v \text{ ramifies in } H_\chi\} \cup S_\infty \subseteq \Sigma$$
$$\Sigma'(\chi) := \{v \nmid p : v \text{ ramifies in } H_\chi\} \cup T \subseteq \Sigma'$$

Lemma 4.3 (Replacing Σ, Σ' by the corresponding sets for H_χ)

Suppose $\mathrm{Sel}_{\Sigma(\chi)}^{\Sigma'(\chi)}(H_\chi)_R$ quadratically presented. Then so is $\mathrm{Sel}_\Sigma^{\Sigma'}(H_\chi)_R$.

$$\mathrm{Fitt}\left(\mathrm{Sel}_\Sigma^{\Sigma'}(H_\chi)_R\right) = \mathrm{Fitt}\left(\mathrm{Sel}_{\Sigma(\chi)}^{\Sigma'(\chi)}(H_\chi)_R\right) \prod_{v\in\Sigma-\Sigma(\chi)} (1-\sigma_v) \prod_{v\in\Sigma'-\Sigma'(\chi)} (1-\sigma_v \mathrm{N}v)$$

$$\mathrm{Fitt} = \textstyle\prod_{v\in\Sigma'-\Sigma'(\chi)}(1-\sigma_v\mathrm{N}v)$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \prod_{v\in\Sigma'-\Sigma'(\chi)}\prod_{w\mid v}(\mathcal{O}_H/w)_R^{\times,\vee} & \longrightarrow & \mathrm{Ker} & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_{H_\chi,S'-\Sigma}\otimes R & \longrightarrow & \mathrm{Hom}_{\mathcal{O}}\left(\left(\mathcal{O}^\times_{H_\chi,S',\Sigma'}\otimes\mathcal{O}\right)^{G'=\chi},\mathcal{O}\right) & \longrightarrow & \mathrm{Sel}_\Sigma^{\Sigma'}(H_\chi)_R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Y_{H_\chi,S'-\Sigma(\chi)}\otimes R & \to & \mathrm{Hom}_{\mathcal{O}}\left(\left(\mathcal{O}^\times_{H_\chi,S',\Sigma'(\chi)}\otimes\mathcal{O}\right)^{G'=\chi},\mathcal{O}\right) & \to & \mathrm{Sel}_{\Sigma(\chi)}^{\Sigma'(\chi)}(H_\chi)_R & \to & 0 \\
 & & \twoheaddownarrow & & \downarrow & & \downarrow & & \\
 & \to & Y_{H_\chi,\Sigma-\Sigma(\chi)}\otimes R & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

$$\mathrm{Fitt} = \textstyle\prod_{v\in\Sigma-\Sigma(\chi)}(1-\sigma_v)$$

Compute the Fitting ideal of $\prod_{v\in\Sigma'-\Sigma'(\chi)}\prod_{w|v}(\mathcal{O}_H/w)_R^{\times,\vee}$ locally.

- w_χ := the place of H_χ below w
- $G_\chi := \mathrm{Gal}(H_\chi/F)$
- $G_{\chi,w_\chi} \simeq \mathrm{Gal}\left((\mathcal{O}_H/w) / \left(\mathcal{O}_{H_\chi}/w_\chi\right)\right) = \langle\sigma_v\rangle$ because w_χ unramified
- ξ := generator of $\left(\mathcal{O}_{H_\chi}/w_\chi\right)^\times$; $\sigma_v(\xi) = \xi^{\mathrm{N}v}$

We have a presentation

$$\begin{array}{ccccc}\mathbb{Z}_p[G_{\chi,w_\chi}] & \hookrightarrow & \mathbb{Z}_p[G_{\chi,w_\chi}] & \twoheadrightarrow & \left(\mathcal{O}_{H_\chi}/w_\chi\right)^\times \otimes \mathbb{Z}_p \\ 1 & \longmapsto & 1-\sigma_v^{-1}\mathrm{N}v & \longmapsto & \xi - \sigma_v^{-1}(\xi^{\mathrm{N}v}) \\ & & 1 & \longmapsto & \xi\end{array}$$

Exactness at $\mathbb{Z}_p[G_{\chi,w_\chi}]$:

$$\#\,\mathrm{Coker} = \prod_{\psi\in G_\chi}\left(1-\psi(\sigma_v^{-1}\mathrm{N}v)\right) = \prod_{i=0}^{\#G_\chi}\left(1-\zeta_{\#G_\chi}^i\mathrm{N}v\right) = \mathrm{N}v^{\#G_\chi}-1 = \#(\mathcal{O}_{H_\chi}/w_\chi)^\times$$

$$\Rightarrow \mathrm{Fitt}_{\mathbb{Z}_p[G_\chi]}\left(\left(\mathcal{O}_{H_\chi}/w_\chi\right)^{\times,\vee}\right) = (1-\sigma_v\mathrm{N}v)$$

□

Removing characters with trivial zeros

Recall: $\Psi \subseteq \widehat{G}$ subset of characters

$$\begin{aligned} \mathcal{O}[G] &\to \textstyle\prod_{\psi\in\Psi} \mathcal{O}_\psi \\ x &\mapsto (\psi(x))_{\psi\in\Psi} \end{aligned} \qquad R_\Psi := \text{image}$$

$\chi \in \widehat{G'}$ $\quad \Psi_\chi := \left\{\psi \in \widehat{G} : \psi|_{G'} = \chi\right\}$ characters belonging to χ $\quad R := R_\chi := R_{\Psi_\chi}$

Trivial zero

If for $\psi \in \Psi_\chi$, there is $v \in \Sigma$ with $\psi(v) = 1$ then $(1 - \psi(v)\mathrm{N}v^{-0}) = 1 - 1 = 0$.
Hence $L_{\Sigma,\Sigma'}(\psi, 0) = 0$, so $\Theta^{\#}_{\Sigma,\Sigma'}$ is a zero-divisor in R.

$$\Psi := \left\{\psi \in \widehat{G} : \psi|_{G'} = \chi, \forall v \in \Sigma : \psi(v) \neq 1\right\} \quad \text{characters without trivial zeros}$$

$$\Psi' := \left\{\psi \in \widehat{G} : \psi|_{G'} = \chi, \exists v \in \Sigma : \psi(v) = 1\right\} \quad \text{characters with a trivial zero}$$

Lemma 7.1 (Removing characters with trivial zeros)

If $\mathrm{Fitt}_{R_\Psi}\left(\mathrm{Sel}_\Sigma^{\Sigma'}(H) \otimes R_\Psi\right) \subseteq \Theta^{\#}_{\Sigma,\Sigma'} R_\Psi$ then $\mathrm{Fitt}_{R_\chi}\left(\mathrm{Sel}_\Sigma^{\Sigma'}(H) \otimes R_\chi\right) \subseteq \Theta^{\#}_{\Sigma,\Sigma'} R_\chi$.

$\approx$ Lemma 3.2

For $\psi \in \Psi'$, i.e. when $\psi(v) = 1$ for some $v \in \Sigma$, then $\mathrm{Fitt}_{\mathcal{O}_\psi}\left(\mathrm{Sel}_\Sigma^{\Sigma'}(H)_\psi\right) = 0$.

Suffices to show: $\mathrm{Sel}_\Sigma^{\Sigma'}(H)_\psi$ is infinite (finitely generated over $\mathcal{O}_\psi$ and Fitt $\subseteq$ Ann)

$X_{H,\Sigma} \hookrightarrow Y_{H,\Sigma} \twoheadrightarrow \mathbb{Z}$ $\qquad$ $K := \mathrm{Frac}\,\mathcal{O}_\psi$

Lemma (Burns–Kurihara–Sano)

$$0 \to \mathrm{Cl}_\Sigma^{\Sigma'}(H)^\vee \to \mathrm{Sel}_\Sigma^{\Sigma'}(H) \to \mathrm{Hom}_\mathbb{Z}\left(\mathcal{O}_{H,\Sigma,\Sigma'}^\times, \mathbb{Z}\right) \to 0$$

Proof: similar to previous one.

$$\begin{aligned} \mathrm{Sel}_\Sigma^{\Sigma'}(H)_\psi \otimes_{\mathcal{O}_\psi} K &\simeq \mathrm{Hom}_\mathbb{Z}\left(\mathcal{O}_{H,\Sigma,\Sigma'}^\times, K\right)_\psi && \text{Lemma} \\ &\simeq \mathrm{Hom}_\mathbb{Z}\left(X_{H,\Sigma}, K\right)_\psi && \text{Dirichlet unit theorem} \\ &\simeq \mathrm{Hom}_K\left((X_{H,\Sigma} \otimes_\mathbb{Z} K)^\psi, K\right) \end{aligned}$$

Suffices to show: $(X_{H,\Sigma} \otimes_\mathbb{Z} K)^\psi \neq 0$

$$(X_{H,\Sigma} \otimes_\mathbb{Z} K)^\psi \simeq (Y_{H,\Sigma} \otimes_\mathbb{Z} K)^\psi \supseteq \left(Y_{H,\{v\}} \otimes_\mathbb{Z} K\right)^\psi = \left(\mathrm{Ind}_{G_v}^G K\right)^\psi \simeq K \neq 0$$

$\mathbb{Z} \otimes_\mathbb{Z} K = K = K^{\mathbb{1}} \oplus \bigoplus_{\psi \neq \mathbb{1}} 0$

$\langle \mathrm{Ind}_{G_v}^G K, K(\psi)\rangle_G = \langle K, \mathrm{Res}_{G_v}^G K(\psi)\rangle_{G_v} = \langle K, K\rangle_{G_v} = 1$ $\square$

$$\Psi := \left\{\psi \in \widehat{G} : \psi|_{G'} = \chi, \forall v \in \Sigma : \psi(v) \neq 1\right\} \quad \text{characters without trivial zeros}$$

$$\Psi' := \left\{\psi \in \widehat{G} : \psi|_{G'} = \chi, \exists v \in \Sigma : \psi(v) = 1\right\} \quad \text{characters with a trivial zero}$$

Lemma 7.1 (Removing characters with trivial zeros)

If $\mathrm{Fitt}_{R_\Psi}\left(\mathrm{Sel}_\Sigma^{\Sigma'}(H) \otimes R_\Psi\right) \subseteq \Theta^{\#}_{\Sigma,\Sigma'} R_\Psi$ then $\mathrm{Fitt}_{R_\chi}\left(\mathrm{Sel}_\Sigma^{\Sigma'}(H) \otimes R_\chi\right) \subseteq \Theta^{\#}_{\Sigma,\Sigma'} R_\chi$.

$$\begin{array}{ccccc}
 & & R = R_{\Psi_\chi} & \hookrightarrow & \prod_{\psi \in \Psi_\chi} \mathcal{O}_\psi \\
 & \nearrow\!\!\!\!\!\nearrow & \Big\downarrow{\scriptstyle \iota = (\iota_1, \iota_2)} & & \Big\| \\
\mathcal{O}[G] & & & & \\
 & \searrow & & & \\
 & & R_\Psi \times R_{\Psi'} & \hookrightarrow & \prod_{\psi \in \Psi} \mathcal{O}_\psi \times \prod_{\psi' \in \Psi'} \mathcal{O}_{\psi'}
\end{array}$$

Later: $\mathrm{Sel}_\Sigma^{\Sigma'}(H)_R$ quadratically presented $\Rightarrow \mathrm{Fitt}_R\left(\mathrm{Sel}_\Sigma^{\Sigma'}(H)_R\right) = (x) \subseteq R$

$\mathrm{Fitt}_{R_\Psi}\left(\mathrm{Sel}_\Sigma^{\Sigma'}(H) \otimes R_\Psi\right) = \iota_1(x) \Rightarrow \iota_1(x) = y \cdot \iota_1\left(\Theta^{\#}_{\Sigma,\Sigma'}\right)$, $y \in R_\Psi$

Lemma 3.2 $\Rightarrow \iota_2(x) R_{\Psi'} = \mathrm{Fitt}_{R_{\Psi'}}\left(\mathrm{Sel}_\Sigma^{\Sigma'}(H) \otimes R_{\Psi'}\right) = 0$

$\iota_2\left(\Theta^{\#}_{\Sigma,\Sigma'}\right) = 0$ by definition

Let $\tilde{y} \in R$ be a lift of y. Then $\iota_{1,2}\left(x - \tilde{y}\Theta^{\#}_{\Sigma,\Sigma'}\right) = 0$ $\square$

Summary: reduction steps

Lemma 4.1	We can remove (non-ramifying) primes above p from Σ'.
Lemmata 4.2 & 4.3	We may assume $\chi \in \widehat{G'}$ to be faithful.
Lemma 7.1	We may disregard characters with trivial zeros.

Part II

The Ritter–Weiss module

Properties of the Ritter–Weiss module

We will construct a $\mathbb{Z}[G]$-module $\nabla_{\Sigma}^{\Sigma'}$ with the following properties:

(P1) $0 \to \mathrm{Cl}_{\Sigma}^{\Sigma'}(H) \to \nabla_{\Sigma}^{\Sigma'} \to X_{H,\Sigma} \to 0$ where $X_{H,\Sigma} = \mathrm{Ker}(Y_{H,\Sigma} \xrightarrow{\text{aug}} \mathbb{Z})$

(P2) $0 \to \mathrm{Cl}_{\Sigma}^{\Sigma'}(H)^{-} \to \nabla_{\Sigma}^{\Sigma',-} \to X_{H,\Sigma}^{-} \to 0$ has extension class in $\mathrm{Ext}^1_{\mathbb{Z}[G]^-}\left(X_{H,\Sigma}^{-}, \mathrm{Cl}_{\Sigma}^{\Sigma'}(H)^{-}\right)$ given by CFT (more on this later)

(P3) $\left(\nabla_{\Sigma}^{\Sigma'}\right)_p^{\mathrm{tr}} \simeq \mathrm{Sel}_{\Sigma}^{\Sigma'}(H)_p$ as $\mathbb{Z}_p[G]$-modules where tr is Jannsen's transpose

(P4) $\left(\nabla_{\Sigma}^{\Sigma'}\right)_p$ is quadratically presented as a $\mathbb{Z}_p[G]$-module

Jannsen's transpose

M a $\mathbb{Z}[G]$-module

$M^* := \mathrm{Hom}_{\mathbb{Z}[G]}(M, \mathbb{Z}[G])$ with contragredient $\mathbb{Z}[G]$-action: $(r \cdot \varphi)(x) := \varphi(r^\# x)$

$P_0 \to P_1 \twoheadrightarrow M$ projective presentation $\Rightarrow P_0^*$, P_1^* are projective

$P_1^* \to P_0^* \twoheadrightarrow M^{\mathrm{tr}}$, well-defined up to homotopy

$\Psi^\# := \{\psi^{-1} : \psi \in \Psi\}$, $R^\# := R_{\Psi^\#}$

Lemma 6.1

If M is a quadratically presented R-module, then M^{tr} is a quadratically presented $R^\#$-module, and $\mathrm{Fitt}_{R^\#}(M^{\mathrm{tr}}) = \mathrm{Fitt}_R(M)^\#$

$$\begin{array}{ccccc} P_0 & \xrightarrow{(a_{i,j})} & P_1 & \longrightarrow\!\!\!\!\!\rightarrow & M \\ P_1^* & \xrightarrow{(a_{j,i}^\#)} & P_0^* & \longrightarrow\!\!\!\!\!\rightarrow & M^{\mathrm{tr}} \end{array}$$

Application

(P3) $\left(\nabla_{\Sigma}^{\Sigma'}\right)_p^{\mathrm{tr}} \simeq \mathrm{Sel}_{\Sigma}^{\Sigma'}(H)_p$ as $\mathbb{Z}_p[G]$-modules

(P4) $\left(\nabla_{\Sigma}^{\Sigma'}\right)_p$ is quadratically presented as a $\mathbb{Z}_p[G]$-module

Lemma 6.1 $\mathrm{Fitt}_{R^\#}(M^{\mathrm{tr}}) = \mathrm{Fitt}_R(M)^\#$

Corollary 6.2

$\mathrm{Sel}_{\Sigma}^{\Sigma'}(H)_R$ is quadratically presented as an R-module, and

$$\mathrm{Fitt}_R\left(\mathrm{Sel}_{\Sigma}^{\Sigma'}(H)_R\right) = \mathrm{Fitt}_{R^\#}\left(\nabla_{\Sigma}^{\Sigma'}(H)_{R^\#}\right)^\# \qquad \square$$

Construction of $\nabla_\Sigma^{\Sigma'}$, step 1

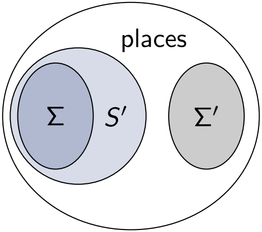
Choose S' auxiliary set: finite set of primes of F such that:

$S' \supseteq \Sigma$ $\qquad$ $S' \cap \Sigma' = \varnothing$ $\qquad$ $\mathrm{Cl}_{S'}^{\Sigma'}(H) = 1$ $\qquad$ $\bigcup_{w \in S'_H} G_w = G$

For all v, fix $w \mid v$. We have V_w, W_w $\mathbb{Z}[G]$-modules such that

$$\forall w : 0 \to H_w^\times \to V_w \to \Delta G_w \to 0$$
$$\forall w \text{ finite} : 0 \to \mathcal{O}_w^\times \to V_w \to W_w \to 0$$

	$v \notin S' \cup \Sigma'$	$v \in S' - \Sigma$	$v \in \Sigma$	$v \in \Sigma'$
J	$\mathcal{O}_w^\times$	$\mathcal{O}_w^\times$	$H_w^\times$	U_w
$\hookdownarrow$	$\Vert$	$\hookdownarrow$	$\hookdownarrow$	$\Vert$
V	$\mathcal{O}_w^\times$	V_w	V_w	U_w
$\twoheaddownarrow$		$\twoheaddownarrow$	$\twoheaddownarrow$	
W		W_w	ΔG_w	



Formally: $J := \widetilde{\prod_{v \notin \Sigma \cup \Sigma'}} \mathcal{O}_w^\times \widetilde{\prod_{v \in \Sigma}} H_w^\times \widetilde{\prod_{v \in \Sigma'}} U_w$ where $\widetilde{\prod_v} M_w := \prod_v \mathrm{Ind}_{G_w}^G M_w$

Construction of $\nabla_{\Sigma}^{\Sigma'}$, step 2

Global CFT: $H^2(G, C_H)$ is cyclic where $C_H := \mathbb{A}_H^\times / H$ idèle class group

$$\begin{array}{ccccc} H^2(G, C_H) & \overset{\sim}{=\!=} & \mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z}, C_H) & \overset{\sim}{=\!=\!=} & \mathrm{Ext}^1(\Delta G, C_H) \\ \text{generator} & \longleftrightarrow & & & [C_H \hookrightarrow \mathfrak{D} \twoheadrightarrow \Delta G] \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\ & & \downarrow \vartheta_J & & \color{red}{\twoheaddownarrow \vartheta} & & \twoheaddownarrow \vartheta_W & & \\ 0 & \longrightarrow & C_H & \longrightarrow & \mathfrak{D} & \longrightarrow & \Delta G & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J' & \longrightarrow & V & \longrightarrow & W' & \longrightarrow & 0 \\ & & \twoheaddownarrow \vartheta_{J'} & & \twoheaddownarrow \vartheta & & \twoheaddownarrow \vartheta_{W'} & & \\ 0 & \longrightarrow & C_H & \longrightarrow & \mathfrak{D} & \longrightarrow & \Delta G & \longrightarrow & 0 \end{array}$$

	$v \notin S' \cup \Sigma'$	$v \in S'$	$v \in \Sigma'$
J'	$\mathcal{O}_w^\times$	$H_w^\times$	U_w
$\hookdownarrow$	$\Vert$	$\hookdownarrow$	$\Vert$
V	$\mathcal{O}_w^\times$	V_w	U_w
$\twoheaddownarrow$		$\twoheaddownarrow$	
W'		ΔG_w	

Verify surjectivity of ϑ:

$\operatorname{Coker} \vartheta_{J'} = \mathrm{Cl}_{S'}^{\Sigma'} = 1$

$\mathcal{G} := \{g \in G : g - 1 \in \operatorname{Im} \vartheta_{W'}\}$,
$G = \bigcup_{v \in S'} G_v \subseteq \mathcal{G} \subseteq G \Rightarrow \vartheta_{W'}$ surjective

Snake lemma: $0 \to \mathcal{O}_{H,\Sigma,\Sigma'}^\times \to V^\vartheta \to W^\vartheta \to \mathrm{Cl}_{\Sigma}^{\Sigma'}(H) \to 0$

Construction of $\nabla_{\Sigma}^{\Sigma'}$, step 3

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W_w & \xrightarrow{(\alpha_w,\beta_w)} & \mathbb{Z}[G_w]^2 & \longrightarrow & W_w^\times & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_w} & & \downarrow\!\!\!\downarrow & & \downarrow & & \\
0 & \longrightarrow & \Delta G_w & \longrightarrow & \mathbb{Z}[G_w] & \xrightarrow{\text{aug}} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

$$\begin{array}{rcl}
W & = & \widetilde{\prod_{v\in\Sigma}} \Delta G_w \quad \widetilde{\prod_{v\in S'-\Sigma}} W_w \\
{\scriptstyle \gamma}\downarrow & & \quad\quad \hookdownarrow \quad\quad\quad\quad \downarrow{\scriptstyle \beta_w} \\
B & := & \prod_{v\in\Sigma} \mathbb{Z}[G] \quad \prod_{v\in S'-\Sigma} \mathbb{Z}[G]
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W & \xrightarrow{\gamma} & B & \longrightarrow & Y_{H,\Sigma} & \longrightarrow & 0 \\
 & & \downarrow\!\!\!\downarrow{\scriptstyle \vartheta_W} & & \downarrow{\scriptstyle \vartheta_B} & & \downarrow{\scriptstyle \vartheta_Y} & & \\
0 & \longrightarrow & \Delta G & \longrightarrow & \mathbb{Z}[G] & \xrightarrow{\text{aug}} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

ϑ_W: as before, surjective
ϑ_B: id on Σ, $(\sigma_v - 1)$ on $S' - \Sigma$
ϑ_Y: augmentation on Σ

Snake lemma: $\quad 0 \to W^\vartheta \to B^\vartheta \to X_{H,\Sigma} \to 0$

Construction of $\nabla_\Sigma^{\Sigma'}$: definition and (P1)

$$0 \to \mathcal{O}_{H,\Sigma,\Sigma'}^\times \to V^\vartheta \to W^\vartheta \to \mathrm{Cl}_\Sigma^{\Sigma'}(H) \to 0$$
$$0 \to W^\vartheta \to B^\vartheta \to X_{H,\Sigma} \to 0$$

Definition (Ritter–Weiss module)

$$\nabla_\Sigma^{\Sigma'} := \mathrm{Coker}\left(V^\vartheta \to W^\vartheta \to B^\vartheta\right)$$

We immediately get:

$$0 \to \mathcal{O}_{H,\Sigma,\Sigma'}^\times \to V^\vartheta \to B^\vartheta \to \nabla_\Sigma^{\Sigma'} \to 0$$
$$0 \to \mathrm{Cl}_\Sigma^{\Sigma'}(H) \to \nabla_\Sigma^{\Sigma'} \to X_{H,\Sigma} \to 0 \qquad \text{(P1)}$$

Projectivity of V^ϑ and property (P4)

Lemma A.4 (Projectivity)

$V_R^\vartheta := V^\vartheta \otimes_{\mathbb{Z}[G]} R$ is projective.

$$0 \to V^\vartheta \to V \to \mathfrak{O} \to 0$$

$\mathfrak{O}$ is cohomologically trivial. Hence V c.t. $\Leftrightarrow$ V^ϑ c.t.

$$V := \widetilde{\prod_{v \notin S' \cup \Sigma'}} \mathcal{O}_w^\times \widetilde{\prod_{v \in S'}} V_w \widetilde{\prod_{v \in \Sigma'}} U_w$$

will show: every factor is c.t.

$\operatorname{Ind}_{G_w}^G \mathcal{O}_w^\times$: higher unit filtration, $H^q(G, U_w^i/U_w^{i+1}) = 0$ using w unramified

$\operatorname{Ind}_{G_w}^G V_w$: c.t. shown in Ritter–Weiss, nothing changes

$\operatorname{Ind}_{G_w}^G U_w$: Want to show U_w is G_v-c.t. As for $\operatorname{Ind}_{G_w}^G \mathcal{O}_w^\times$: $U_w^{I_w}$ is G_w/I_w-c.t.
Inf-Res argument: suffices to show that U_w is I_w-c.t.
$\exists U_w' \subseteq U_w$ c.t. open subgroup.
$(U_w : U_w')$ is a power of the residue characteristic $\neq p$ by Lemma 4.1
$\Rightarrow (U_w : U_w') \in R^\times \Rightarrow (U_w)_R \simeq (U_w')_R$

V^ϑ is $\mathbb{Z}$-torsion free: $\mathcal{O}_{H,\Sigma,\Sigma'}^\times$, B^ϑ are and $0 \to \mathcal{O}_{H,\Sigma,\Sigma'}^\times \to V^\vartheta \to B^\vartheta \to \nabla_\Sigma^{\Sigma'} \to 0$

Nakayama: fin. generated $\mathbb{Z}$-torsion free G-module is c.t. iff $\mathbb{Z}[G]$-projective $\square$

Lemma A.4 (Quadratic presentation)

$V_R^\vartheta \to B_R^\vartheta \twoheadrightarrow \nabla_\Sigma^{\Sigma'}(H)_R$ is a quadratic presentation.

Recall: $$0 \to B^\vartheta \to B := \prod_{v \in S'} \mathbb{Z}[G] \xrightarrow{\vartheta_B} \mathbb{Z}[G] \to 0$$

Suffices to show: $\forall \psi \colon \mathbb{Z}[G] \to \overline{\mathbb{Q}}^\times$:

$$\dim_{\overline{\mathbb{Q}}} V_\chi^\vartheta := \dim_{\overline{\mathbb{Q}}} V^\vartheta \otimes_{\mathbb{Z}[G]} \overline{\mathbb{Q}}(\chi) = \dim_{\overline{\mathbb{Q}}} B_\chi^\vartheta$$

$\overline{\mathbb{Q}}(\chi)$ is flat over $\mathbb{Z}[G]$

$$0 \to \mathrm{Cl}_\Sigma^{\Sigma'}(H) \to \nabla_\Sigma^{\Sigma'} \to X_{H,\Sigma} \to 0$$

$$\Rightarrow \dim_{\overline{\mathbb{Q}}} \left(\nabla_\Sigma^{\Sigma'}\right)_\chi = \dim_{\overline{\mathbb{Q}}} (X_{H,\Sigma})_\chi \underset{\text{unit thm}}{\overset{\text{Dirichlet}}{=}} \dim_{\overline{\mathbb{Q}}} \left(\mathcal{O}_{H,\Sigma,\Sigma'}^\times\right)_\chi$$

$$0 \to \mathcal{O}_{H,\Sigma,\Sigma'}^\times \to V^\vartheta \to B^\vartheta \to \nabla_\Sigma^{\Sigma'} \to 0$$

$$\Rightarrow \dim_{\overline{\mathbb{Q}}} V_\chi^\vartheta = \dim_{\overline{\mathbb{Q}}} B_\chi^\vartheta$$

□

Property (P3)

Lemma A.8 (P3)

$$\nabla_{\Sigma}^{\Sigma'}(H)_R^{\mathrm{tr}} \simeq \mathrm{Sel}_{\Sigma}^{\Sigma'}(H)_R$$

$$\mathcal{F} := \mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(-,\mathbb{Z}[G]) =: (-)^* : \mathbb{Z}[G]\text{-Mod} \to \mathbb{Z}[G]\text{-Mod}$$

$$\varphi \mapsto \left(m \mapsto \textstyle\sum_{g\in G} \varphi(gm)[g^{-1}]\right)$$

$$0 \to W^\vartheta \to B^\vartheta \to X_{H,\Sigma} \to 0 \quad \overset{\mathcal{F}}{\rightsquigarrow} \quad \mathcal{F}(B^\vartheta) \twoheadrightarrow \mathcal{F}(W^\vartheta)$$

$$0 \to \mathcal{O}_{H,\Sigma,\Sigma'}^\times \to V^\vartheta \to W^\vartheta \to \mathrm{Cl}_{\Sigma}^{\Sigma'}(H) \to 0 \quad \overset{\mathcal{F}}{\rightsquigarrow} \quad \mathcal{F}(W^\vartheta) \hookrightarrow \mathcal{F}(V^\vartheta)$$

$$\begin{array}{ccc} \mathcal{F}(B^\vartheta) & \longrightarrow & \mathcal{F}(V^\vartheta) \twoheadrightarrow \left(\nabla_{\Sigma}^{\Sigma'}\right)^{\mathrm{tr}} \\ & \searrow\!\!\!\!\searrow \quad \nearrow & \\ & \mathcal{F}(W^\vartheta) & \end{array}$$

$$\mathbb{Z} \hookrightarrow W_w \xrightarrow{\alpha_w} \Delta G_w \quad \overset{\widetilde{\prod}_{v\in S'-\Sigma}}{\rightsquigarrow} \quad Y_{H,S'-\Sigma} \hookrightarrow W^\vartheta \twoheadrightarrow (W')^\vartheta$$

$$\overset{\mathcal{F}}{\rightsquigarrow} \quad \mathcal{F}((W')^\vartheta) \hookrightarrow \mathcal{F}(W^\vartheta) \twoheadrightarrow \mathcal{F}(Y_{H,S'-\Sigma})$$

$$
\begin{array}{ccccc}
J' & \hookrightarrow & V & \twoheadrightarrow & W' \\
\Big\downarrow{\scriptstyle \vartheta_{J'}} & & \Big\downarrow{\scriptstyle \vartheta} & & \Big\downarrow{\scriptstyle \vartheta_{W'}} \\
C_H & \hookrightarrow & \mathfrak{O} & \twoheadrightarrow & \Delta G
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{O}^{\times}_{H,S',\Sigma} & \hookrightarrow & V^{\vartheta} & \twoheadrightarrow & (W')^{\vartheta} \\
 & & \big\downarrow{\scriptstyle \mathcal{F}} & & \\
\mathcal{F}\left((W')^{\vartheta}\right) & \hookrightarrow & \mathcal{F}\left(V^{\vartheta}\right) & \twoheadrightarrow & \mathcal{F}\left(\mathcal{O}^{\times}_{H,S',\Sigma}\right)
\end{array}
$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Ker} & \xrightarrow{\sim} & \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{F}\left((W')^{\vartheta}\right) & \longrightarrow & \mathcal{F}\left(V^{\vartheta}\right) & \longrightarrow & \mathcal{F}\left(\mathcal{O}^{\times}_{H,S',\Sigma'}\right) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\| & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{F}\left(W^{\vartheta}\right) & \longrightarrow & \mathcal{F}\left(V^{\vartheta}\right) & \longrightarrow & \left(\nabla^{\Sigma'}_{\Sigma}\right)^{\mathrm{tr}} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & \xrightarrow{\sim} & \mathcal{F}\left(Y_{H,S'-\Sigma}\right) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
$$

Recall: $$0 \to \prod_{v \in S'_H - \Sigma_H} \mathbb{Z} \to \mathrm{Hom}_{\mathbb{Z}}\left(\mathcal{O}^{\times}_{H,S',\Sigma'}, \mathbb{Z}\right) \to \mathrm{Sel}^{\Sigma'}_{\Sigma}(H) \to 0$$ $\square$

Property (P2): extension class via CFT

$\mathrm{rec}_{L/H} : M := \mathrm{Cl}_{\Sigma}^{\Sigma'}(H)^{-} \xrightarrow{\sim} \mathrm{Gal}(L/H)$ Artin reciprocity

Lemma 6.6: for any $\mathbb{Z}[G]^{-}$-module N, $\mathrm{res}_{G_H}^{G_F} : H^1(G_F, N) \xrightarrow{\sim} H^1(G_H, N)^G$

$\varpi : G_H \to \mathrm{Gal}(L/H) \xrightarrow{\mathrm{rec}_{L/H}^{-1}} M$ $\qquad\qquad$ $\varpi \in \mathrm{Hom}_{\mathrm{cts}}(G_H, M) = H^1(G_H, M)$

Lemma 6.6 $\Rightarrow \exists! \lambda \in H^1(G_F, M) : \mathrm{res}_{G_H}^{G_F}(\lambda) = \varpi$

$$\begin{array}{ccccc} H^1(G_v, M)^{G_{F,v}} & \xhookrightarrow{\mathrm{inf}} & H^1(G_F, M) & \xrightarrow{\mathrm{res}_{G_{F,v}}^{G_F}} & H^1(G_{F,v}, M)^{G_v} \\ \exists! \lambda_v & \longmapsto & \lambda & \longmapsto & 0 \end{array}$$

$$\lambda|_{G_{H,v}} = \lambda|_{G_{F,v} \cap G_H} = \varpi|_{G_{H,w}} = 0$$

$$\begin{aligned} \mathrm{Ext}^1_{\mathbb{Z}[G]^-}\left(X_{H,\Sigma}^{-}, M\right) &\simeq \bigoplus_{v \in \Sigma} \mathrm{Ext}^1_{\mathbb{Z}[G]^-}\left(\left(\mathrm{Ind}_{G_v}^{G} \mathbb{Z}\right)^{-}, M\right) \\ &\simeq \bigoplus_{v \in \Sigma} \mathrm{Ext}^1_{\mathbb{Z}[\frac{1}{2}][G_v]}\left(\mathbb{Z}\left[\tfrac{1}{2}\right], M\right) \\ &\simeq \bigoplus_{v \in \Sigma} H^1(G_v, M) \qquad \ni (\lambda_v)_{v \in \Sigma} \end{aligned}$$

We make the isomorphism $\mathrm{Ext}^1_{\mathbb{Z}[G]^-}\left(X^-_{H,\Sigma}, M\right) \simeq \bigoplus_{v\in\Sigma} H^1(G_v, M)$ explicit:

$$0 \longrightarrow M = \mathrm{Cl}^{\Sigma'}_{\Sigma}(H)^- \longrightarrow \nabla^{\Sigma',-}_{\Sigma} \longrightarrow X^-_{H,\Sigma} \longrightarrow 0$$

$$x \longmapsto \tfrac{1}{2}\left(w - \overline{w}\right)$$

$$\gamma_v : G \to M \qquad g \mapsto gx - x$$

$[\gamma_v] \in H^1(G_v, M)$ independent of the choice of x

Lemma 6.7/A.9 (P2)

$$(\lambda_v)_{v\in\Sigma} = (\gamma_v)_{v\in\Sigma}$$

Suffices to show: $\exists x : \lambda_v = \gamma_v$

We may assume $\exists v' \in S' : \sigma_{v'} = c =$ complex conjugation

Then by definition, ϑ_B at v' is multiplication by $(c-1)$.

$\tilde{x} \in B^{\vartheta,-} \subseteq \prod_{S'} \mathbb{Z}[G]^-$: $\frac{1-c}{2}$ at v, $\frac{1}{2}$ at v', 0 elsewhere

$\forall g \in G_v : \gamma_v(g) = (g-1)\tilde{x} = \frac{(1-c)(g-1)}{2}$ at v, $\frac{g-1}{2}$ at v', 0 elsewhere

Compute its image under the snake map $W^\vartheta \to M$, described explicitly by Ritter–Weiss. □

Thank you for your attention!